CHARACTERIZATION OF FUNDAMENTAL
SOLUTIONS TO GENERALIZED PELL EQUATIONS

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ABSTRACT. We give a characterization of fundamental solutions to
generalized Pell equations. We discuss other literature, especially
Nagell.

1. Introduction

This article has two main goals:

• A characterization of fundamental solutions to generalized Pell
equations.
• A comparison to other literature, especially Nagell.

The next section defines fundamental solutions to generalized Pell
equations and gives the main theorem that characterizes those solu-
tions. Following sections step back and take things from the beginning,
building up to a self-contained proof of the characterization.

Misreadings of Nagell’s theorems, especially Theorem 109 [3], have
caused endless confusion in the literature. I hope this note helps reduce
some of that confusion.

2. Definition and characterization of fundamental
solutions to generalized Pell equations

Throughout $D$ will be positive non-square integer and $t + u\sqrt{D}$ will
denote the minimal positive solution to $t^2 - Du^2 = 1$ in integers $t, u$.
Note that $t \geq 2$ and $u \geq 1$. Also $t - u\sqrt{D} = 1/(t + u\sqrt{D})$ whence

\begin{equation}
0 < t - u\sqrt{D} < 1.
\end{equation}

We consider the generalized Pell equation

\begin{equation}
x^2 - Dy^2 = N
\end{equation}

\textit{Date:} October 4, 2014.
for \( N \) an integer, \( N \neq 0, 1 \).

If this equation has a solution, it has infinitely many solutions. These solutions can be separated into a finite number of equivalence classes using the following equivalence relation.

**Definition 1.** Two solutions \( x + y\sqrt{D} \) and \( x' + y'\sqrt{D} \) to (2) are equivalent if there is an integer \( n \) and a sign \( \pm 1 \) so that

\[
(3) \quad x' + y'\sqrt{D} = \pm(x + y\sqrt{D})(t + u\sqrt{D})^n.
\]

It is not hard to show that two solutions \( x + y\sqrt{D} \) and \( x' + y'\sqrt{D} \) to (2) are equivalent if and only if \( xx' \equiv yy' \pmod{N} \) and \( xy' \equiv x'y \pmod{N} \). See Appendix A for a proof of this statement and related discussion.

We define fundamental solutions of (2) as follows:

**Definition 2.** If \( N > 1 \), the fundamental solution in an equivalence class of solutions to (2) is the solution with the minimal positive \( x \). If there are two equivalent solutions with the same minimal positive \( x \), then the solution with \( y > 0 \) is the fundamental solution.

If \( N < 0 \), the fundamental solution in an equivalence class of solutions to (2) is the solution with the minimal positive \( y \). If there are two equivalent solutions with the same minimal positive \( y \), then the solution with \( x > 0 \) is the fundamental solution.

This is DeLeon’s [1] definition of fundamental solution, which differs somewhat from Nagell’s [3] definition. Definitions of fundamental solution other than those of DeLeon and Nagell are possible.

We now show that this is a valid definition. First consider the case \( N > 1 \). If a class has a solution \( x + y\sqrt{D} \) with \( x < 0 \), then another solution in the class is \( -x - y\sqrt{D} \), for which \( -x > 0 \). No solution has \( 0 \leq x < \sqrt{N} \) (else \( y^2 = (x^2 - N)/D < 0 \)). So every class of solutions has at least one solution with \( x \geq \sqrt{N} \). Because the \( x \) are integers, there is a least positive such \( x \). If \( x = \sqrt{N} \) then \( y = 0 \), so there is at most one solution with \( x = \sqrt{N} \). For any \( x > \sqrt{N} \), there are only two possible values for \( y \), namely \( y = \pm\sqrt{(x^2 - N)/D} \), and of these exactly one is positive.
If $N < 0$, then every class has a solution with $y > 0$, and no solution has $0 \leq y < \sqrt{|N|/D}$. Thus every class has a solution with $y \geq \sqrt{|N|/D}$. If $y = \sqrt{|N|/D}$, then $x = 0$. If $y > \sqrt{|N|/D}$, then the only two possible values for $x$ are $x = \pm \sqrt{Dy^2 + N}$, and of these two, exactly one is positive.

The following theorem characterizes fundamental solutions to (2).

**Theorem 1.** (A) If $N > 1$, then $x + y\sqrt{D}$ satisfying (2) is a fundamental solution if and only if one of the following hold:

(i) $\sqrt{N} \leq x < \sqrt{\frac{1}{2}N(t+1)}$,

(ii) $x = \sqrt{\frac{1}{2}N(t+1)}$ and $y = u\sqrt{\frac{|N|}{2(t-1)}}$.

(B) If $N < 0$, then $x + y\sqrt{D}$ satisfying (2) is a fundamental solution if and only if one of the following hold:

(i) $\sqrt{|N|/D} \leq y < u\sqrt{\frac{|N|}{2(t-1)}}$,

(ii) $y = u\sqrt{\frac{|N|}{2(t-1)}}$ and $x = \sqrt{\frac{1}{2}|N|(t-1)}$.

For $N > 1$, this theorem is illustrated in Figure 1, using the equation $x^2 - 3y^2 = 6$. Here $D = 3$, $N = 6$, $t = 2$, and $u = 1$. Solutions to (2) in real numbers lie on a hyperbola. If $N > 1$, then the $x$-axis is the axis of the hyperbola, and if $N < 0$, then the $y$-axis is the axis of the hyperbola. In Figure 1 the lighter lines represent all real solutions to $x^2 - 3y^2 = 6$. The heavier lines represent the region where fundamental solutions might lie. The filled in circle emphasizes that the point $x = 3$, $y = 1$ is a fundamental solution. The open circle at $x = 3$, $y = -1$ emphasizes that this point is not a fundamental solution (it is equivalent to $(3,1)$).

For $N < 0$, Figure 2 illustrates the theorem using the equation $x^2 - 3y^2 = -8$. Similar to the previous graph, $D = 3$, $N = -8$, $t = 2$, and $u = 1$. The lighter lines represent real solutions and the heavier lines represent the region where fundamental solutions might lie. The filled in circle emphasizes that the point $x = 2$, $y = 2$ is a fundamental solution. The open circle at $x = -2$, $y = 2$ emphasizes that this point is not a fundamental solution (being equivalent to $(2,2)$).
Figure 1. Region of fundamental solutions

There are many other ways to state Theorem 1. The formulas given in Appendix B help translate among equivalent statements of Theorem 1. The bounds used in Theorem 1 (and in Appendix B) are derived in Appendix C. But, note that our proof of Theorem 1 shows that these bounds are correct.

3. Structure of solutions to generalized Pell equations

Before getting into the technicalities of a proof of Theorem 1, let’s take an informal graphical look at the structure of solutions to generalized Pell equations.

We begin by just plotting all of the integer solutions to \( x^2 - 162y^2 = 81 \) for which \( 0 < x < e^{50} \) in Figure 3. Because as \( x \) gets larger, integer solutions are farther and farther apart, we plot these solutions on a “log scale” in Figure 3 (if a coordinate \( z \) is negative, we use \( -\log(-z) \), and we have fudged the point \((9, 0)\) because log of zero is not defined).

Using separate markers for each equivalence class in Figure 4, we see that there are 6 classes of solutions to this equation. Note how regularly they occur—between any two “consecutive” solutions in any one class, exactly one solution from each of the other 5 classes occurs, and these
are always in the same order, as you follow the hyperbola counterclockwise, starting at the lower right. (I think the relatively even spacing between different classes is not typical.) Values of coordinates of points plotted are given in two tables in Appendix F—Tables 1 and 2. The fundamental solutions are $(9, 0), (27, \pm 2), (153, \pm 12),$ and $(891, 70).$ The minimal positive $t, u$ so that $t^2 - 162u^2 = 1$ are $t = 19601, u = 1540.$

Let’s drill down on just one class, the “squares.” Figure 5 plots the solutions with $x > 0$ for the class with fundamental solution $x = 153, y = -12.$ You can check that the fundamental solution for this class satisfies Theorem 1. We have labeled the point $(153, -12)$ as “$P_0.$” Other points are labeled “$P_i$” so that

$$P_i = P_0(t + u\sqrt{D})^i = (153 - 12\sqrt{162})(19601 + 1540\sqrt{162})^i.$$ 

We can use Figure 5 to see the result of the “action” of $t + u\sqrt{D}$ on a solution to a generalized Pell equation. Applying $t + u\sqrt{D}$ to the
point $P_{-4}$ gives the point $P_{-3}$. Continuing to apply $t + u\sqrt{D}$, gives the points $P_{-2}$, $P_{-1}$, \ldots, $P_4$, and so on. Applying $t - u\sqrt{D}$ takes us through the points in the reverse order.

Returning to Figure 4, let me point out the two points $(891, 70)$ and $(891, -70)$. These points are marked by diamonds, and are on the same vertical line ($\log 891 \approx 6.8$). These points are equivalent $(891 + 70\sqrt{162} = (891 - 70\sqrt{162})(19601 + 1540\sqrt{162}))$, and satisfy the upper bound equalities given in Theorem 1. That is,

$$891 = \sqrt{\frac{1}{2} \cdot 81 \cdot 19602},$$

as is easily checked.

Imagine that there were a solution just a scootch to the right of the point $(891, -70)$ (there isn’t, but suppose there were). Applying $t + u\sqrt{D}$ to this hypothetical point would give a point a scootch to the
left of (891, 70). The point to the right of (891, −70) would not satisfy the bounds given in Theorem 1 (|x| > 891), but the point to the left of (891, 70) does satisfy the bounds (|x| < 891). Conversely, if there were a point a scotch to the left of (891, −70), it would satisfy the bounds given in Theorem 1, while the point generated by applying $t + u \sqrt{D}$ to this point would be just to the right of (891, 70), and would not satisfy the bounds given in Theorem 1.

This illustrates how moving a point $P$ just a bit along the curve moves the point $P(t + u \sqrt{D})$ just a bit. In fact $P(t + u \sqrt{D})$ is a continuous “function” of $P$. More importantly, the example illustrates that the upper bounds in Theorem 1 occur (when $N > 1$) at the point $(x, y)$ with $x, y > 0$ exactly when $x + y \sqrt{D} = (x - y \sqrt{D})(t + u \sqrt{D})$. (When $N < 0$, the upper bounds occur for the point $(x, y)$ with $x, y > 0$ exactly when $x + y \sqrt{D} = (-x + y \sqrt{D})(t + u \sqrt{D})$.)
Figure 5. One class of solutions to \( x^2 - 162y^2 = 81 \)

Figure 6 is similar to Figure 4, except that it is for the equation \( x^2 - 384y^2 = -384 \), for which \( N < 0 \). Here there are four classes of solutions, occurring in a regular pattern similar to that for the equation \( x^2 - 162y^2 = 81 \). The fundamental solutions are \((0, 1), (96, \pm 5), \) and \((960, 49)\). The minimal positive \( t, u \) so that \( t^2 - 384y^2 = 1 \) are \( t = 4801, u = 245 \). The values plotted are given in Tables 3 and 4 in Appendix F.

We have not illustrated any points on the branches of the hyperbolas that are in Quadrants II and III when \( N > 1 \) and Quadrants III and IV when \( N < 0 \). Each such point is the negative of a point for the branches for which we have given examples. You can see that applying \( t + u\sqrt{D} \) to points on these (neglected) branches moves along the curve in a clockwise direction.
Figure 6. Solutions to $x^2 - 384y^2 = -384$

4. SOME BACKGROUND

Some lemmas that will be useful follow.

Lemma 1. (A) If $N > 1$, $x = \sqrt{N(t + 1)/2}$, and $y = u\sqrt{N}/(2(t + 1))$, then

$$x + y\sqrt{D} = (x - y\sqrt{D})(t + u\sqrt{D}).$$

(B) If $N < 0$, $x = \sqrt{|N|(t - 1)/2}$, and $y = u\sqrt{|N|}/(2(t - 1))$, then

$$x + y\sqrt{D} = (-x + y\sqrt{D})(t + u\sqrt{D}).$$

Proof. This is easily established by direct calculation. □

Lemma 2. If $x + y\sqrt{D}$ and $x' + y'\sqrt{D}$ are solutions to (2), then

$$|x| \lesssim |x'| \iff |y| \lesssim |y'|.$$
Proof. This follows from $|x| = \sqrt{Dy^2 + N}$ and $|y| = \sqrt{(x^2 - N)/D}$. □

Lemma 3. (A) If $N > 1$, $x > 0$, and $x' + y'\sqrt{D} = (x + y\sqrt{D})(t + u\sqrt{D})$, then $x' > 0$ and $y' > y$.
(B) If $N > 1$, $x > 0$, and $x' + y'\sqrt{D} = (x + y\sqrt{D})(t - u\sqrt{D})$, then $x' > 0$ and $y' < y$.
(C) If $N < 0$, $y > 0$, and $x' + y'\sqrt{D} = (x + y\sqrt{D})(t + u\sqrt{D})$, then $y' > 0$ and $x' > x$.
(D) If $N < 0$, $y > 0$, and $x' + y'\sqrt{D} = (x + y\sqrt{D})(t - u\sqrt{D})$, then $y' > 0$ and $x' < x$.

This lemma formalizes some of our earlier comments about the relation between $x + y\sqrt{D}$ and $(x + y\sqrt{D})(t + u + \sqrt{D})$. For example, for $N > 1$ and $x > 0$, Lemma 3 says that applying $t + u\sqrt{D}$ moves points counter-clockwise around the branch of the hyperbola.

Proof. We begin with (A) and first we show that $x' > 0$. Regardless of the sign of $y$, $x' = xt + yu\sqrt{D} \geq xt - |y|uD$. But

$$xt - |y|uD = t(x - |y|\sqrt{D}) + |y|\sqrt{D}(t - u\sqrt{D}) > 0$$

by (1) and because $x - |y|\sqrt{D} = N/(x + |y|\sqrt{D}) > 0$.

Next, we have $y' = xu + yt$. If $y \geq 0$ then $yt \geq y$, so $y' = xu + yt > yt \geq y$.

Now assume that $y < 0$. Then

$$y' = xu + yt = u\sqrt{D}|y|^2 + N - t|y|.$$ We need to show that the right-hand side of this equation is greater than $y = -|y|$.

From (1) we have $t - u\sqrt{D} < 1$, so $u\sqrt{D} > t - 1$, whence

$$\frac{u}{t - 1} > \frac{1}{\sqrt{D}}.$$ It is easy to see that

$$\frac{1}{\sqrt{D}} > \frac{1}{\sqrt{D + N/|y|^2}} = \frac{|y|}{\sqrt{D}|y|^2 + N}.$$
Therefore
\[ \frac{u}{t-1} > \frac{1}{\sqrt{D}} > \frac{|y|}{\sqrt{D}|y|^2 + N} \]
and so \( u\sqrt{D}|y|^2 + N > (t-1)|y| \), which gives \( u\sqrt{D}|y|^2 + N - t|y| > -|y| \), showing that \( y' > y \).

For (B), to see that \( x' > 0 \), as in the proof of (A) we have
\[ x' = xt - yu\sqrt{D} \geq xt - |y||u\sqrt{D} > 0 \]
where the last inequality is from (6).

For \( y' < y \), we just note that \((x' + y'\sqrt{D})(t + u\sqrt{D}) = x + y\sqrt{D} \), so \( y' < y \) follows from (A).

Now we turn to (C). Here
\[ y' = xu + yt \geq yt - |x|u = y(t - u\sqrt{D}) + u(y\sqrt{D} - |x|) > 0 \]
by (1) and because \( y\sqrt{D} - |x| = -N/(y\sqrt{D} + |x|) > 0 \).

Next, \( x' = xt + yuD \). If \( x \geq 0 \), then \( xt \geq x \), whence \( x' = xt + yuD > xt \geq x \).

Now we assume that \( x < 0 \). We want to show that
\[ -|x|t + yuD > -|x|, \]
or, equivalently, that
\[ yuD > (t - 1)|x|. \]
Now
\[ \frac{\sqrt{D}|y|^2 - |N|}{yD} = \sqrt{1 - |N|/D|y|^2} < \frac{1}{\sqrt{D}}, \]
so
\[ \frac{|x|}{yD} < \frac{1}{\sqrt{D}} \]
because \( |x| = \sqrt{D|y|^2 + N} = \sqrt{D|y|^2 - |N|} \). From (7) we have
\[ \frac{|x|}{yD} < \frac{u}{t-1} \]
from which (9) follows.

For (D), \( y' = yt - xu \geq yt - |x|u > 0 \) where the last inequality is from (8).

That \( x' < x \) follows from (C) by considering \((x' + y'\sqrt{D})(t + u\sqrt{D}) = x + y\sqrt{D} \). \( \square \)
Lemma 4. (A) If $N > 1$, $x_1$, $x_2 > 0$, $y_2 > y_1$,

(i) $x_1' + y_1' \sqrt{D} = (x_1 + y_1)(t + u \sqrt{D})$, and

(ii) $x_2' + y_2' \sqrt{D} = (x_2 + y_2)(t + u \sqrt{D})$,

then $y_2' > y_1'$.

(B) If $N < 0$, $y_1y_2 > 0$, $x_2 > x_1$,

(i) $x_1' + y_1' \sqrt{D} = (x_1 + y_1)(t + u \sqrt{D})$, and

(ii) $x_2' + y_2' \sqrt{D} = (x_2 + y_2)(t + u \sqrt{D})$,

then $x_2' > x_1'$.

This says that the “order” of solutions is preserved by the action of $t + u \sqrt{D}$.

Proof. Let $x' + y' \sqrt{D} = (x + y \sqrt{D})(t + u \sqrt{D})$. To establish (A), it suffices to show that $dy'/dy > 0$. Because $x > 0$, we have

$$y' = xu + ty = u \sqrt{Dy^2 + N} + ty,$$

so

$$\frac{dy'}{dy} = \frac{uyD}{\sqrt{Dy^2 + N}} + t.$$

If $y > 0$, both terms are positive, and we’re done. Now assume $y < 0$. We need to show that

$$t - \frac{u|y|D}{\sqrt{D|y|^2 + N}} > 0.$$

But

$$t - \frac{u|y|D}{\sqrt{D|y|^2 + N}} = t - \frac{u \sqrt{D}}{\sqrt{1 + N/(D|y|^2)}} > t - u \sqrt{D} > 0.$$

completing the proof for (A).

For (B),

$$x' = xt + yuD = xt + u \sqrt{D(x^2 + |N|)}$$
so
\[
\frac{dx'}{dx} = t + \frac{uxD}{\sqrt{D(x^2 + |N|)}}
\geq t - \frac{|x|D}{\sqrt{D(x^2 + |N|)}}
= t - \frac{u\sqrt{D}}{\sqrt{1 + |N|/(Dx^2)}}
> t - u\sqrt{D} > 0.
\]

Lemma 5. If \(x + y\sqrt{D}\) and \(x' + y'\sqrt{D}\) are equivalent solutions to (2), then

\[uN | xy' - x'y.\]

Proof. Let \(t_n + u_n\sqrt{D} = (t + u\sqrt{D})^n\) for some integer \(n \geq 0\). If \(x + y\sqrt{D}\) and \(x' + y'\sqrt{D}\) are equivalent solutions to (2), then for some \(n\) and some choice of a pair of signs \(\epsilon_t, \epsilon_u = \pm 1,\)

\[x' + y'\sqrt{D} = (x + y\sqrt{D})(\epsilon_t t_n + \epsilon_u u_n\sqrt{D}).\]

Then
\[x' = \epsilon_t x t_n + \epsilon_u y u_n D,\]
and
\[y' = \epsilon_u x u_n + \epsilon_t y t_n.\]

Then
\[xy' - x'y = u_x x^2 u_n + \epsilon_t x y t_n - \epsilon_t x y t_n - \epsilon_u y^2 u_n D = \epsilon_u u_n(x^2 - y^2 D) = \epsilon_u u_n N.\]

But \(u|u_n \ [2, \text{Cor. to Thm. 3, p. 70}], \) so \(uN | xy' - x'y.\)

As an aside, similarly, \(t_n N = \epsilon_t (xx' - yy' D).\) The only positive integer that divides all \(t_n\) is 1 (in fact \(t_2 = 2t^2 + 1, \) so \(\gcd(t_1, t_2) = 1),\) so the strongest statement corresponding to that in Lemma 5 is that \(N|xx' - yy' D.\)
5. Proof of Theorem 1

We begin with $N > 1$. First we show that every fundamental solution satisfies Theorem 1(A). Suppose $x + y\sqrt{D}$ is the solution with minimal $x > 0$. If $x > \sqrt{N(t+1)/2}$, then $|y| > u\sqrt{N/(2(t+1))}$ by the fact that $(\sqrt{N(t+1)/2}, \pm u\sqrt{N/(2(t+1))})$ are real points on (2) and Lemma 2. If $y > \sqrt{N/(2(t+1))}$, then for the equivalent solution $x' + y'\sqrt{D} = (x + y\sqrt{D})(t - u\sqrt{D})$, we have

(i) $x' > 0$ by Lemma 3(B),
(ii) $y' < y$ by Lemma 3(B), and
(iii) $y' > -\sqrt{N/(2(t+1))}$ by Lemma 1(A) and Lemma 4,
so $|y'| < |y|$. But then $x' < x$ and so $x + y\sqrt{D}$ is not the fundamental solution for the class.

Similarly, if $y < -u\sqrt{N/(2(t+1))}$, then for $x' + y'\sqrt{D} = (x + y\sqrt{D})(t + u\sqrt{D})$, we have $x' > 0$, $y' > y$, and $y' < u\sqrt{N/(2(t+1))}$. Then $|y'| < |y|$, so $x' < x$.

If $y = -u\sqrt{N/(2(t+1))}$, then $x = \sqrt{N(t+1)/2}$, and $x + y\sqrt{D}$ is equivalent to $x - y\sqrt{D}$. This last solution has $-y > 0$, and so is the fundamental solution. Hence every fundamental solution satisfies Theorem 1(A).

To show that every solution satisfying Theorem 1(A) is a fundamental solution, it is sufficient to show that no two solutions satisfying Theorem 1(A) are equivalent. Suppose $x + y\sqrt{D}$ and $x' + y'\sqrt{D}$ are distinct solutions to (2) that satisfy Theorem 1(A). Then

\begin{align}
\text{(10)} & \quad \sqrt{N} \leq x, x' \leq \sqrt{\frac{1}{2}N(t+1)}, \\
\text{(11)} & \quad -u\sqrt{\frac{N}{2(t+1)}} < y, y' \leq u\sqrt{\frac{N}{2(t+1)}}.
\end{align}

Now,

$$|xy'| < \left(\sqrt{\frac{1}{2}N(t+1)}\right) \left(\sqrt{\frac{N}{2(t+1)}}\right) = uN/2$$

because if $x = \sqrt{N(t+1)/2}$, then $|y'| < u\sqrt{N/(2(t+1))}$. 
Similarly, \(|x'y'| < uN/2\). Also, \(xy' - x'y \neq 0\) because \(xy' - x'y = 0\) implies that \(y/x = y'/x'\), which is impossible for distinct points on the same branch of the hyperbola (2).

We have
\[0 < |xy' - x'y| \leq |xy'| + |x'y| < \frac{uN}{2} + \frac{uN}{2} = uN.\]

Thus \(uN\) does not divide \(|xy' - x'y|\), and by Lemma 5, the solutions \(x + y\sqrt{D}\) and \(x' + y'\sqrt{D}\) are not equivalent.

Now consider \(N < 0\). If \(x + y\sqrt{D}\) is a solution to (2) with \(x > 0\) and \(y > u\sqrt{|N|/(2(t - 1))}\), then the equivalent solution \(x' + y'\sqrt{D} = (x + y\sqrt{D})(t - u\sqrt{D})\) has \(0 < y' < y\). If \(x + y\sqrt{D}\) is a solution to (2) with \(x < 0\) and \(y > u\sqrt{|N|/(2(t - 1))}\), then the equivalent solution \(x' + y'\sqrt{D} = (x + y\sqrt{D})(t + u\sqrt{D})\) has \(0 < y' < y\). If \(x = -\sqrt{|N|(t - 1)/2}\) and \(y = u\sqrt{|N|/(2(t - 1))}\), then \(x + y\sqrt{D}\) is equivalent to \(-x + y\sqrt{D} = (x + y\sqrt{D})(t + u\sqrt{D})\). Thus, every fundamental solution satisfies Theorem 1(B).

Suppose \(x + y\sqrt{D}\) and \(x' + y'\sqrt{D}\) are distinct solutions to (2) that satisfy Theorem 1(B). Then
\[
\begin{align*}
|1|N(t-1)/2 &< x, x' \leq \sqrt{\frac{1}{2}N(t-1)}, \\
\sqrt{|N|/D} &< y, y' \leq u\sqrt{\frac{N}{2(t-1)}}. 
\end{align*}
\]

Now,
\[
|x'y'| < \left(\sqrt{\frac{1}{2}N(t-1)}\right)\left(u\sqrt{\frac{N}{2(t-1)}}\right) = uN/2
\]

because if \(y = u\sqrt{N/(2(t - 1))}\), then \(|x'| < \sqrt{N(t - 1)/2}\).

Similarly, \(|xy'| < uN/2\). Also, \(xy' - x'y \neq 0\), as above. We have
\[0 < |xy' - x'y| \leq |xy'| + |x'y| < \frac{uN}{2} + \frac{uN}{2} = uN.\]

Thus \(uN\) does not divide \(|xy' - x'y|\), and by Lemma 5, the solutions \(x + y\sqrt{D}\) and \(x' + y'\sqrt{D}\) are not equivalent.
6. Comparison to Nagell

Nagell’s definition of a fundamental solution to (2) for \( N \neq 0, 1 \), (actually, Nagell does not exclude \( N = 1 \), but it is good to do so) can be written as follows [3]:

**Definition 3.** A solution \( x + y\sqrt{D} \) to (2) is the fundamental solution in its class if \( y \) is the minimal non-negative \( y \) among all solutions in the class. If two solutions in the class have the same minimal non-negative \( y \), then the solution with \( x > 0 \) is the fundamental solution.

The following theorem characterizes fundamental solutions under Nagell’s definition.

**Theorem 2.** (A) If \( N > 1 \), then \( x + y\sqrt{D} \) satisfying (2) is a fundamental solution if and only if one of the following hold:
   
   (i) \( 0 < y < u\sqrt{\frac{N}{2(t+1)}} \),
   
   (ii) \( y = 0 \) and \( x = \sqrt{N} \),
   
   (iii) \( y = u\sqrt{\frac{N}{2(t+1)}} \) and \( x = \sqrt{\frac{N(t+1)}{2}} \).

(B) If \( N < 0 \), then \( x + y\sqrt{D} \) satisfying (2) is a fundamental solution if and only if one of the following hold:

   (i) \( \sqrt{\frac{|N|}{D}} \leq y < u\sqrt{\frac{|N|}{2(t-1)}} \),
   
   (ii) \( y = u\sqrt{\frac{|N|}{2(t-1)}} \) and \( x = \sqrt{\frac{|N|(t-1)}{2}} \).

As this theorem shows, for \( N < 0 \), Nagell’s definition coincides with DeLeon’s. For \( N > 1 \),

- If \( x > 0 \) and \( y \geq 0 \), then \( x + y\sqrt{D} \) is a either fundamental solution under both the DeLeon or Nagell definitions or is not a fundamental solution under either.

- If \( xy < 0 \) and \( x + y\sqrt{D} \) is a fundamental solution under one definition, then \( -x - y\sqrt{D} \) is a fundamental solution under the other definition.

Figure 7 illustrates the region where fundamental solutions might lie under Nagell’s definition for the equation \( x^2 - 3y^2 = 6 \), for which \( N > 0 \). Note that the region is disconnected, and one must be careful to include the point \( (\sqrt{N}, 0) \) among the fundamental solutions (when \( N \) is a square), and exclude the point \( (-\sqrt{N}, 0) \).

References

7. Appendix A—Characterization of equivalent solutions

We prove that \(x + y\sqrt{D}\) and \(x' + y'\sqrt{D}\) are equivalent if and only if \(N|xx' - yy'D\) and \(N|xy' - x'y\), and a bit more.

For \(n \geq 0\), let \(t_n + u_n\sqrt{D} = (t + u\sqrt{D})^n\). Note that \(t_n - u_n\sqrt{D} = (t + u\sqrt{D})^{-n}\). If \(x + y\sqrt{D}\) and \(x' + y'\sqrt{D}\) are equivalent solutions to \(x^2 + Dy^2 = N\), then for some \(n \geq 0\) and some \(\epsilon_t, \epsilon_u = \pm 1\), \(x' + y'\sqrt{D} = (x + y\sqrt{D})(\epsilon_t t_n + \epsilon_u u_n\sqrt{D})\).

**Lemma 6.** Let \(x + y\sqrt{D}\) and \(x' + y'\sqrt{D}\) be solutions to \(x^2 + Dy^2 = N\). Then

(A) If \(x + y\sqrt{D}\) and \(x' + y'\sqrt{D}\) are equivalent with \(x' + y'\sqrt{D} = (x + y\sqrt{D})(\epsilon_t t_n + \epsilon_u u_n\sqrt{D})\), then

   (i) \(xx' - yy'D = \epsilon_t t_n N\),
   (ii) \(xy' - x'y = \epsilon_u u_n N\),
   (iii) \(N|xx' - yy'D\),
   (iv) \(uN|xy' - x'y\).

(B) If \(xx' - yy'D = aN\) and \(xy' - x'y = bN\), then

   (i) \(a^2 - Db^2 = 1\),
   (ii) There are \(n \geq 0\), \(\epsilon_t, \epsilon_u = \pm 1\), so that \(a = \epsilon_t t_n\) and \(b = \epsilon_u u_n\),
   (iii) \(x' + y'\sqrt{D} = (x + y\sqrt{D})(a + b\sqrt{D})\),
   (iv) \(x + y\sqrt{D}\) and \(x' + y'\sqrt{D}\) are equivalent.

(C) \(x + y\sqrt{D}\) and \(x' + y'\sqrt{D}\) are equivalent if and only if \(N|xx' - yy'D\) and \(N|xy' - x'y\).

**Proof.** To be added (it is very straightforward). \(\square\)
8. Appendix B—Some useful formulas

We give some formulas that are useful for working with solutions to (2). These are really just properties of real points on hyperbolas.

If $N > 1$ then

\[
\sqrt{\frac{1}{2}N(t+1)} = u\sqrt{\frac{ND}{2(t-1)}}
\]

(14)

\[
u\sqrt{\frac{N}{2(t+1)}} = \sqrt{\frac{N(t-1)}{2D}}
\]

(15)

\[
\sqrt{N} < |x| < \sqrt{\frac{1}{2}N(t+1)} \iff 0 < |y| < u\sqrt{\frac{N}{2(t+1)}}
\]

(16)

\[
|x| = \sqrt{N} \iff y = 0
\]

(17)

\[
|x| = \sqrt{\frac{1}{2}N(t+1)} \iff |y| = u\sqrt{\frac{N}{2(t+1)}}.
\]

(18)

If $N < 0$ then

\[
\sqrt{\frac{1}{2}|N|(t-1)} = u\sqrt{\frac{|N|D}{2(t+1)}}
\]

(19)

\[
u\sqrt{\frac{|N|}{2(t-1)}} = \sqrt{\frac{|N|(t+1)}{2D}}
\]

(20)

\[
0 < |x| < \sqrt{\frac{1}{2}|N|(t-1)} \iff \sqrt{\frac{|N|}{D}} < |y| < u\sqrt{\frac{|N|}{2(t-1)}}
\]

(21)

\[
x = 0 \iff |y| = \sqrt{\frac{|N|}{D}}
\]

(22)

\[
|x| = \sqrt{\frac{1}{2}|N|(t-1)} \iff |y| = u\sqrt{\frac{|N|}{2(t-1)}}.
\]

(23)
9. Appendix C—Derivation of Nagell’s bounds on fundamental solutions

We derive the bounds on fundamental solutions used in Theorems 1 and 2.

For $D > 0$, $D$ not a square, and $N \neq 0, 1$, we consider the equation

\[(24)\quad x^2 - Dy^2 = N.\]

We take $t + u\sqrt{D}$ to be the minimal positive solution to the equation

\[(25)\quad x^2 - Dy^2 = 1.\]

The case $N > 0$

For $N > 0$ we want $x + y\sqrt{D}$, $x > 0$, $y > 0$ so that $x + y\sqrt{D}$ is a solution to (24) and

\[(26)\quad x + y\sqrt{D} = (x - y\sqrt{D})(t + u\sqrt{D}).\]

Expanding the right-hand side of (26) and equating rational and irrational parts gives:

\[(27)\quad x = xt - yuD,\]
\[(28)\quad y = xu - yt.\]

Rearranging (28) gives

\[(29)\quad y(t + 1) = xu.\]

Substituting $x = \sqrt{Dy^2 + N}$ from (24) gives

\[(30)\quad y(t + 1) = u\sqrt{Dy^2 + N}.\]

Squaring and rearranging gives

\[(31)\quad y^2((t + 1)^2 - Du^2) = u^2N.\]

Plugging in $t^2 - 1$ for $Du^2$ gives

\[(32)\quad y^2((t + 1)^2 - (t^2 - 1)) = u^2N.\]

Expanding $(t+1)^2 - (t^2 - 1)$ gives $(t+1)^2 - (t^2 - 1) = 2t + 2 = 2(t+1)$.

Whence, from (32):

\[(33)\quad y^2(2(t + 1)) = u^2N,\]
or

\[ y = u \sqrt{\frac{N}{2(t + 1)}}. \]  

Substituting the right-hand side of (34) for \( y \) in (29) and solving for \( x \) gives

\[ x = \sqrt{\frac{N(t + 1)}{2}}. \]  

Note that solving for \( x \) and \( y \) from just equations (27) and (28) does not work. Each of these follows from the other using (25).

**The case \( N < 0 \)**

For \( N < 0 \) we want \( x + y\sqrt{D}, x > 0, y > 0 \) so that so that \( x + y\sqrt{D} \)

is a solution to (24) and

\[ x + y\sqrt{D} = (-x + y\sqrt{D})(t + u\sqrt{D}). \]  

Expanding the right-hand side of (36) and equating rational and irrational parts gives:

\[ x = -xt + yuD, \]  

(37) \hspace{1cm} y = -xu + yt.  

(38)

Rearranging (38) gives

\[ y(t - 1) = xu. \]  

(39)

Substituting \( x = \sqrt{Dy^2 - |N|} \) from (24) gives

\[ y(t - 1) = u\sqrt{Dy^2 - |N|}. \]  

(40)

Squaring (40), substituting \( t^2 - 1 \) for \( Du^2 \), and solving for \( y^2 \) gives

\[ y^2 = \frac{u^2|N|}{2(t - 1)}, \]  

whence

\[ y = u\sqrt{\frac{|N|}{2(t - 1)}}. \]  

(42)

Formulas for \( x \) are easily derived from the formulas for \( y \).
This appendix states Nagell’s theorems [3] verbatim. [Eventually I might add the complete Section 58 through the statement of Theorem 109, without the proofs. That would then include Nagell’s definition of equivalence as he stated it.]

Nagell’s theorems [3] are quite carefully stated, but are frequently misread. Nagell gives necessary conditions for a solution to (2) to be a fundamental solution, but does not give sufficient conditions, and never claims to give sufficient conditions.

Numbering of equations in this section is taken from Nagell [3, pp. 204–208]. One equation Nagell refers to in his Theorems 108, 108a, and 109 is:

\[ x^2 - Dy^2 = 1. \]  

In Nagell’s notation, the fundamental solution to this equation \( x_1 + y_1 \sqrt{D} \) is the least positive solution (this differs from the definition of fundamental solution to (2)).

For Theorems 108 and 108a Nagell is assuming that \( D > 0 \) is not a square, and that \( N > 0 \). Note that under the definition of fundamental solution for the generalized Pell equation (2), the fundamental solution of equation (2) is \( 1 + 0 \cdot \sqrt{D} \). As just noted, usually the fundamental solution for (2) is taken to be that with minimal positive \( x \) and \( y \), not minimal non-negative \( y \).

Here are Nagell’s theorems 108, 108a, and 109, as stated by Nagell [3, pp. 204–208].

**Theorem 108.** If \( u + v \sqrt{D} \) is the fundamental solution of the class \( K \) of the equation

\[ u^2 - Dv^2 = N, \]

and if \( x_1 + y_1 \sqrt{D} \) is the fundamental solution of equation (2), we have the inequalities

\[ 0 \leq v \leq \frac{y_1}{\sqrt{2(x_1+1)}} \cdot \sqrt{N}, \]

\[ 0 < |u| \leq \sqrt{\frac{1}{2}(x_1 + 1)N}. \]
Theorem 108a. If \( u + v\sqrt{D} \) is the fundamental solution of the class \( K \) of the equation
\[ u^2 - Du^2 = -N, \]
and if \( x_1 + y_1\sqrt{D} \) is the fundamental solution of equation (2), we have the inequalities
\[
0 < v \leq \frac{y_1}{\sqrt{2(x_1-1)}} \cdot \sqrt{N},
\]
\[
0 \leq |u| \leq \sqrt{\frac{1}{2}(x_1-1)N}.
\]

Theorem 109. If \( D \) and \( N \) are natural numbers, and if \( D \) is not a perfect square, the Diophantine equations (3) and (8) have a finite number of classes of solutions. The fundamental solutions of all the classes can be found after a finite number of trials by means of the inequalities in Theorems 108 and 108a. If \( u^* + v^*\sqrt{D} \) is the fundamental solution of the class \( K \) we obtain all the solutions \( u + v\sqrt{D} \) of \( K \) by the formula
\[
\begin{align*}
  u + v\sqrt{D} &= (u^* + v^*\sqrt{D})(x + y\sqrt{D}),
\end{align*}
\]
where \( x + y\sqrt{D} \) runs through all the solutions of equation (2), including \( \pm 1 \).

The Diophantine equation (3), or (8), has no solutions at all when it has no solution satisfying the inequalities (4) and (5), or (9) and (10), respectively.
11. Appendix E—Counterexamples to Tsangaris’ “converse” of Nagell’s theorems


**Theorem 1.1** The Diophantine equation (F) has a finite number of classes of solutions. The fundamental solutions of all such classes are determined by the following (equivalent) inequalities in case $C > 0$

\begin{align*}
(1.3) & \quad 0 < |X^*| \leq \sqrt{(x_1 + 1)C/2}, \\
(1.4) & \quad 0 \leq Y^* \leq (y_1/\sqrt{2(x_1 + 1)})\sqrt{C}
\end{align*}

and by the following (equivalent) inequalities in case $C < 0$

\begin{align*}
(1.5) & \quad 0 \leq |X^*| \leq \sqrt{(x_1 - 1)(-C)/2}, \\
(1.6) & \quad 0 < Y^* \leq (y_1/\sqrt{2(x_1 - 1)})\sqrt{-C}.
\end{align*}

Moreover, $A$ consists of all elements of the form

$$X + Y\sqrt{d} = (X^* + Y^*\sqrt{d})(x + y\sqrt{d})$$

where $x + y\sqrt{d}$ ranges over the set of all integral solutions of (P).

The Diophantine equation (F) has no solution at all when it has no solution satisfying the inequalities (1.3) and (1.4) or (1.5) and (1.6) respectively.

Equation (F) is

\begin{equation}
X^2 - dY^2 = C,
\end{equation}

where $d > 0$ is an integer and not a square, and $C \neq 0$ is an integer.

Equation (P) is

\begin{equation}
x^2 - dy^2 = 1,
\end{equation}

and $x_1 + y_1\sqrt{d}$ is the fundamental solution to (P). The set $A$ is the class of solutions for which $X^* + Y^*\sqrt{d}$ is the fundamental solution.

It is easy to find counterexamples to Tsangaris’s Theorem 1.1.

For the equation $x^2 - 162y^2 = 81$, the solutions $x = -9$, $y = 0$, and $x = -891$, $y = 70$ satisfy the conditions of [4, Thm. 1.1], but are not fundamental solutions. In each case $x + y\sqrt{162}$ is equivalent to $-x + y\sqrt{162}$, and Nagell is quite clear that in these cases the solution with $x > 0$ is the fundamental solution. Here $t = 19601$ and $u = 1540$. These solutions are therefore counterexamples to [4, Thm. 1.1].

For the equation $x^2 - 192y^2 = -192$, the solution $x = -96$, $y = 7$ is a counterexample to [4, Thm. 1.1]. Here $t = 97$ and $u = 7$. 
Tsangaris is essentially claiming that the exact converses to Nagell’s Theorems 108 and 108a are true. Nagell was aware that the exact converses to his Theorems 108 and 108a are not true, as evidenced by his careful wording of his Theorem 109. Tsangaris [4, Thm. 1.1] says, “The fundamental solutions of all such classes are determined by the following (equivalent) inequalities . . . .” In plain English, this says, among other things, that any solution satisfying the inequalities is a fundamental solution for some class. Nagell [3, Thm. 109], on the other hand, says, “The fundamental solutions of all the classes can be found after a finite number of trials by means of the inequalities in Theorems 108 and 108a.” This says that all fundamental solutions are included among the solutions that satisfy the inequalities, but does not say that every solution satisfying the inequalities is in fact a fundamental solution. Indeed, it is not hard to find as many examples as you want of solutions to generalized Pell equations (2) that satisfy the inequalities of [3, Thms. 108 and 108a] or [4, Thm. 1.1] and are not fundamental solutions.
12. Appendix F—Solutions to generalized Pell equations

Tables 1 to 4 list some solutions to two generalized Pell equations discussed in the body of this note.
Solutions to $x^2 - 162y^2 = 81$ with $y < 0$

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Table 1
Solutions to $x^2 - 162y^2 = 81$ with $y \geq 0$

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<tr>
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<td>1.6533E+20</td>
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</tr>
</tbody>
</table>

**Table 2**
Solutions to \( x^2 - 384y^2 = -384 \) with \( x < 0 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>Class</th>
<th>(- \log(-x))</th>
<th>( \log(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-7.8366E+22</td>
<td>3.9991E+21</td>
<td>4</td>
<td>-52.7157</td>
<td>49.7403</td>
</tr>
<tr>
<td>-7.9165E+21</td>
<td>4.0399E+20</td>
<td>2</td>
<td>-50.4232</td>
<td>47.4479</td>
</tr>
<tr>
<td>-7.9973E+20</td>
<td>4.0811E+19</td>
<td>1</td>
<td>-48.1308</td>
<td>45.1555</td>
</tr>
<tr>
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<td>4.1228E+18</td>
<td>3</td>
<td>-45.8384</td>
<td>42.8631</td>
</tr>
<tr>
<td>-8.1614E+18</td>
<td>4.1648E+17</td>
<td>4</td>
<td>-43.5459</td>
<td>40.5706</td>
</tr>
<tr>
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<td>4.2073E+16</td>
<td>2</td>
<td>-41.2535</td>
<td>38.2782</td>
</tr>
<tr>
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<td>4.2503E+15</td>
<td>1</td>
<td>-38.9611</td>
<td>35.9858</td>
</tr>
<tr>
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<td>4.2936E+14</td>
<td>3</td>
<td>-36.6686</td>
<td>33.6933</td>
</tr>
<tr>
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<td>4.3375E+13</td>
<td>4</td>
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<td>31.4009</td>
</tr>
<tr>
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<td>-32.0838</td>
<td>29.1085</td>
</tr>
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<tr>
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<td>4.4716E+10</td>
<td>3</td>
<td>-27.4989</td>
<td>24.5236</td>
</tr>
<tr>
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<td>4.5173E+09</td>
<td>4</td>
<td>-25.2065</td>
<td>22.2312</td>
</tr>
<tr>
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<td>4.5634E+08</td>
<td>2</td>
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<td>19.9387</td>
</tr>
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<td>1</td>
<td>-20.6216</td>
<td>17.6463</td>
</tr>
<tr>
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<td>4,656,965</td>
<td>3</td>
<td>-18.3292</td>
<td>15.3539</td>
</tr>
<tr>
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<td>13.0614</td>
</tr>
<tr>
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<td>2</td>
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</tr>
<tr>
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<td>-11.4519</td>
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<td>6.1841</td>
</tr>
<tr>
<td>-960</td>
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<td>4</td>
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<td>3.8918</td>
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</table>

Table 3
Solutions to $x^2 - 384y^2 = -384$ with $x \geq 0$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>Class</th>
<th>$\log(x)$</th>
<th>$\log(y)$</th>
</tr>
</thead>
<tbody>
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<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
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<td>5</td>
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<td>4.5643</td>
<td>1.6094</td>
</tr>
<tr>
<td>960</td>
<td>49</td>
<td>4</td>
<td>6.8669</td>
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Table 4