FUNDAMENTAL SOLUTIONS TO GENERALIZED PELL EQUATIONS

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Abstract. We characterize what Nagell defines as a fundamental solution to a generalized Pell equation. We give simple counterexamples to theorems in the literature that give incorrect statements of converses of certain of Nagell's theorems.

1. A Characterization of Fundamental Solutions to Generalized Pell Equations

If the Diophantine equation

$$x^2 - Dy^2 = N \tag{1}$$

for $D > 1$, D not a square, $N \neq 0$ has a solution, it has infinitely many solutions. These solutions can be separated into a finite number of equivalence classes using the equivalence relation that considers two solutions (x, y) and (x', y') to be equivalent when there are integers t, u, n, so that

$$x' + y'\sqrt{D} = (x + y\sqrt{D})(t + u\sqrt{D})^n$$

and $t^2 - Du^2 = 1$. Henceforth, we define $t + u\sqrt{D}$ as the least positive solution to $x^2 - Dy^2 = 1$.

For many purposes, it is convenient to identify each class of solutions with a particular element of the class, which is then called the fundamental solution for the class. Picking an element with the smallest magnitude for x and y in the class has a certain appeal.

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Nagell's definition of a fundamental solution to (1) for $N \neq 0, 1$, (actually, Nagell does not exclude $N = 1$, but it is good to do so) can be written as follows [1]:

> A solution $x + y\sqrt{D}$ to (1) is the *fundamental solution* in its class if y is the minimal non-negative y among all solutions in the class. If two solutions in the class have the same minimal non-negative y, then the solution with $x > 0$ is the fundamental solution.

Note that definitions other than Nagell's are possible, but we only use Nagell's definition here.

The following two theorems characterize fundamental solutions to (1).

Theorem 1. *If $N > 1$, then $x + y\sqrt{D}$ satisfying (1) is a fundamental solution if and only if one of the following hold:*

(i) $0 < y < u\sqrt{\frac{N}{2(t+1)}} = \sqrt{\frac{N(t-1)}{2D}}$,

(ii) $y = 0$ *and* $x = \sqrt{N}$,

(iii) $y = u\sqrt{\frac{N}{2(t+1)}}$ *and* $x = \sqrt{\frac{N(t+1)}{2}}$.

Theorem 2. *If $N < 0$, then $x + y\sqrt{D}$ satisfying (1) is a fundamental solution if and only if one of the following hold:*

(i) $\sqrt{\frac{|N|}{D}} \leq y < u\sqrt{\frac{|N|}{2(t-1)}} = \sqrt{\frac{|N|(t+1)}{2D}}$,

(ii) $y = u\sqrt{\frac{|N|}{2(t-1)}}$ *and* $x = \sqrt{\frac{|N|(t-1)}{2}}$.

Note that for $N > 1$, when $0 < y < u\sqrt{\frac{N}{2(t+1)}}$, then

$$\sqrt{N} < |x| < \sqrt{\frac{N(t+1)}{2}} = u\sqrt{\frac{ND}{2(t-1)}},$$

and that for $N < 0$, when $\sqrt{\frac{|N|}{D}} < y < u\sqrt{\frac{|N|}{2(t-1)}}$, then

$$0 < |x| < \sqrt{\frac{|N|(t-1)}{2}} = u\sqrt{\frac{|N|D}{2(t+1)}}.$$

Of course, if $N < 0$ and $y = \sqrt{|N|/D}$ then $x = 0$.

Proofs of these theorems are easy modifications of Nagell's proofs, but with attention to items (ii) and (iii) in Theorem 1 and item (ii) in Theorem 2. These are the cases where $-x + y\sqrt{D}$ is equivalent to the $x + y\sqrt{D}$ specified, so $x + y\sqrt{D}$ is the fundamental solution, yet $-x + y\sqrt{D}$ satisfies Nagell's inequalities. A future version of this note might give self-contained proofs, but such a proof would be based on Nagell's proofs.

2. Some comments

The literature often gets this wrong. For example, it is easy to find counterexamples to Tsangaris's Theorem 1.1 in [2].

For the equation $x^2 - 162y^2 = 81$, the solutions $x = -9$, $y = 0$, and $x = -891$, $y = 70$ satisfy the conditions of [2, Thm. 1.1], but are not fundamental solutions. In each case $x + y\sqrt{162}$ is equivalent to $-x + y\sqrt{162}$, and Nagell is quite clear that in these cases the solution with $x > 0$ is the fundamental solution. Here $t = 19601$ and $u = 1540$. These solutions are therefore counterexamples to [2, Thm. 1.1].

For the equation $x^2 - 192y^2 = -192$, the solution $x = -96$, $y = 7$ is a counterexample to [2, Thm. 1.1]. Here $t = 97$ and $u = 7$.

Tsangaris is essentially claiming that the exact converses to Nagell's Theorems 108 and 108a (reproduced in the section "Nagell's Theorems") are true. Nagell was aware that the exact converses to his Theorems 108 and 108a are not true, as evidenced by his careful wording of his Theorem 109. Tsangaris [2, Thm. 1.1] says, "The fundamental solutions of all such classes are determined by the following (equivalent) inequalities ...." In plain English, this says, among other things, that any solution satisfying the inequalities is a fundamental solution for some class. Nagell [1, Thm. 109], on the other hand, says, "The fundamental solutions of all the classes can be found after a finite number of trials by means of the inequalities in Theorems 108 and 108a." This says that all fundamental solutions are included among the solutions that satisfy the inequalities, but does not say that every solution satisfying the inequalities is in fact a fundamental solution. Indeed, it is not hard to find as many examples as you want of solutions to generalized Pell equations (1) that satisfy the inequalities of [1, Thms. 108 and 108a] or [2, Thm. 1.1] and are not fundamental solutions.

3. Nagell's Theorems

Nagell's theorems [1] are quite carefully stated, but are frequently misread. Nagell gives necessary conditions for a solution to (1) to be a fundamental solution, but does not give sufficient conditions, and never claims to give sufficient conditions.

Numbering of equations in this section is taken from Nagell [1, pp. 204–208]. One equation Nagell refers to in his Theorems 108, 108a, and 109 is:

$$x^2 - Dy^2 = 1. \tag{2}$$

The fundamental solution to *this equation* $x_1+y_1\sqrt{D}$ is the least positive solution (this differs from the definition of fundamental solution to (1)).

For Theorems 108 and 108a Nagell is assuming that $D > 0$ is not a square, and that $N > 0$. Note that under the definition of fundamental solution for the generalized Pell equation (1), the fundamental solution of equation (2) is $1 + 0 \cdot \sqrt{D}$. As just noted, usually the fundamental solution for (2) is taken to be that with minimal positive x and y, not minimal non-negative y.

Here are Nagell's theorems as stated by Nagell [1, pp. 204–208].

Theorem 108. *If $u + v\sqrt{D}$ is the fundamental solution of the class* $\mathbf{K}$ *of the equation*

$$u^2 - Dv^2 = N, \tag{3}$$

and if $x_1 + y_1\sqrt{D}$ is the fundamental solution of equation (2), *we have the inequalities*

$$0 \le v \le \frac{y_1}{\sqrt{2(x_1+1)}} \cdot \sqrt{N}, \tag{4}$$

$$0 < |u| \le \sqrt{\tfrac{1}{2}(x_1+1)N}. \tag{5}$$

Theorem 108a. *If $u + v\sqrt{D}$ is the fundamental solution of the class* $\mathbf{K}$ *of the equation*

$$u^2 - Dv^2 = -N, \tag{8}$$

and if $x_1 + y_1\sqrt{D}$ is the fundamental solution of equation (2), *we have the inequalities*

$$0 < v \leq \frac{y_1}{\sqrt{2(x_1-1)}} \cdot \sqrt{N}, \tag{9}$$

$$0 \leq |u| \leq \sqrt{\tfrac{1}{2}(x_1-1)N}. \tag{10}$$

Theorem 109. *If D and N are natural numbers, and if D is not a perfect square, the Diophantine equations (3) and (8) have a finite number of classes of solutions. The fundamental solutions of all the classes can be found after a finite number of trials by means of the inequalities in Theorems 108 and 108a. If $u^* + v^*\sqrt{D}$ is the fundamental solution of the class $\mathbf{K}$ we obtain all the solutions $u + v\sqrt{D}$ of $\mathbf{K}$ by the formula*

$$u + v\sqrt{D} = (u^* + v^*\sqrt{D})(x + y\sqrt{D}),$$

where $x + y\sqrt{D}$ runs through all the solutions of equation (2), including ± 1.

The Diophantine equation (3), or (8), has no solutions at all when it has no solution satisfying the inequalities (4) and (5), or (9) and (10), respectively.

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References

[1] T. Nagell, *Introduction to Number Theory*, Wiley, New York, 1951.

[2] P. G. Tsangaris, Fermat-Pell equation and the numbers of the form $w^2 + (w+1)^2$, *Publ. Math. Debrecen* **47 / 1–2** (1995), 127–138.

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