Continued Fractions for $\sqrt{D}$ and $(1+\sqrt{D})/2$

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Introduction

For $D > 0$ not a square, $D \equiv 1 \pmod 4$, let L_1 $(= L_1(D))$ and L_4 $(= L_4(D))$ denote the lengths of the periods of the continued fraction expansions of $\sqrt{D}$ and $(1+\sqrt{D})/2$ respectively. This note shows that if $L_4 = 3$ then L_1 cannot be 7. When $L_4 = 3$, literature that I am aware of prohibits any values for L_1 other than 1, 5, 7, 11, and 15, but I have not seen anything showing that 7 is not possible [3, 1]. I would welcome any information as to whether this has been known previously. More generally, I conjecture that if $L_4 \equiv 3 \pmod 6$, then L_1 cannot equal $L_4 + 4$. This last point is not pursued further here.

Outline of proof

We start with an outline of the proof. If $L_4 = 3$, then the continued fraction expansion of $(1+\sqrt{D})/2$ must be of the form $\langle c;\ \overline{a,\ a,\ b}\rangle$ [2, p. 252, exercise 5.3.14]. Then, using the assumption that $D \equiv 1 \pmod 4$, we can show that there must be an $n > 0$ so that $b = a + n(a^2 + 1)$ and that b is odd. If a is odd, n can be any nonnegative integer, while if a is even, n must be an odd positive integer. Then one simply considers four cases: 1) $a = 1$, in which case $L_1 = 1$ (because D will turn out to be $s^2 + 1$ for some integer s), 2) $a > 1$ is odd, whence $L_1 = 5$, 3) $a = 2$, whence $L_1 = 11$, and 4) $a > 2$ is even, whence $L_1 = 15$. In no case is $L_1 = 7$. Proofs of the cases are straightforward: just compute the continued fraction expansion of $\sqrt{D}$ algebraically by the standard method.

Background and notation

First we present the PQa algorithm, which computes the (simple) continued fraction expansion of the quadratic irrational $(P_0 + \sqrt{D})/Q_0$ for certain P_0, Q_0, D. It also computes some auxiliary variables.

Let P_0, Q_0, D be integers so that $Q_0 \neq 0$, $D > 0$ is not a square, and $P_0^2 \equiv D \pmod{|Q_0|}$. Set $A_{-2} = 0$, $A_{-1} = 1$, $B_{-2} = 1$, and $B_{-1} = 0$. For $i \geq 0$ set

$$a_i = \left\lfloor (P_i + \sqrt{D})/Q_i \right\rfloor,$$

$$A_i = a_i A_{i-1} + A_{i-2},$$

$$B_i = a_i B_{i-1} + B_{i-2},$$

and for $i \geq 1$ set

$$P_i = a_{i-1} Q_{i-1} - P_{i-1} \text{ and}$$

$$Q_i = (D - P_i^2)/Q_{i-1}.$$

The (simple) continued fraction expansion of $(P_0+\sqrt{D})/Q_0$ is $\langle a_0, a_1, a_2, \ldots \rangle$ [2, p. 251, exercise 5.3.6]. The convergents to the continued fraction expansion of $(P_0 + \sqrt{D})/Q_0$ are A_i/B_i.

We will only use this with $P_0 = 0$ and $Q_0 = 1$, or, for $D \equiv 1 \pmod 4$, with $P_0 = 1$ and $Q_0 = 2$. In these cases, all of the P_i, Q_i, a_i, A_i, and B_i involved will be positive integers, except possibly P_0.

Also in these cases, the sequences P_i, Q_i, and a_i have certain palindromic properties. Namely, if l is the length of the period, then $P_i = P_{l+1-i}$ for $i = 1, 2, 3, \ldots, l$, $Q_i = Q_{l-i}$ for $i = 0, 1, 2, \ldots, l$, and $a_i = a_{l-i}$ for $i = 1, 2, 3, \ldots, l-1$ [2, p. 242, Corollary 5.3.1; p. 252, exercises 5.3.14 and 5.3.17]. In particular, if $l = 3$ then $a_1 = a_2$.

Full proof

Recall that we assume $D > 0$ is not a square, and $D \equiv 1 \pmod 4$.

As $L_4 = 3$, the continued fraction of $(1+\sqrt{D})/2$ is of the form $\langle c;\ \overline{a,\ a,\ b}\rangle$ for positive integers a, b, and c, $a \neq b$, with the a, a, b repeating, and $b = 2c - 1$. The continued fraction expansion of $\xi = (2b - 2c + 1 + \sqrt{D})/2$ $= (b + \sqrt{D})/2$ is then $\langle b;\ \overline{a,\ a,\ b}\rangle$, which is purely periodic.

Now we establish that $b = a + n(a^2 + 1)$ for some integer $n > 0$, b is odd, and we write D in terms of a and n.

We have $B_2\xi^2 - (A_2 - B_1)\xi - A_1 = 0$ [2, p. 240] where $A_1 = ab + 1$, $A_2 = a^2b + a + b$, $B_1 = a$, and $B_2 = a^2 + 1$. This gives $(a^2 + 1)\xi^2 - (a^2b + b)\xi - (ab + 1) = 0$. Then

$$\xi = \frac{b(a^2 + 1) \pm \sqrt{b^2(a^2 + 1)^2 + 4(a^2 + 1)(ab + 1)}}{2(a^2 + 1)}$$

$$= \frac{b \pm \sqrt{b^2 + \frac{4(ab+1)}{a^2+1}}}{2}$$

The "$\pm$" just above must be "$+$" because $\xi > 0$.

Additionally, we must have $D = b^2 + \frac{4(ab+1)}{a^2+1}$, and so $\frac{4(ab+1)}{a^2+1}$ has to be an integer. In fact, this has to be an even integer, since at most one power of 2 can divide $a^2 + 1$. Because D is odd, b must also be odd.

If $\frac{4(ab+1)}{a^2+1}$ is an integer, then $\frac{4(ab+1)}{a^2+1} - 4 = \frac{4a(b-a)}{a^2+1}$ is an integer, so $a^2 + 1$ divides $4(b - a)$ (as $\gcd(a, a^2 + 1) = 1$). If a is even, then $a^2 + 1$ is odd, and it must be that $a^2 + 1$ divides $(b - a)$. Say this quotient is n. As b is odd and a is even, n must be odd. We can rewrite this as $b = a + n(a^2 + 1)$.

If a is odd, 2 divides $a^2 + 1$ but 4 does not. So $(a^2 + 1)/2$ is odd and divides $b - a$, which is even, so the quotient k is even. Setting $n = k/2$ gives $b = a + n(a^2 + 1)$ for some integer n. There is no restriction on the parity of n when a is odd.

Given that $b = a + n(a^2 + 1)$, we have

$$D = b^2 + 4(an + 1) = n^2a^4 + 2na^3 + (2n^2 + 1)a^2 + 6na + n^2 + 4.$$

Also, note that $n > 0$, because if $n = 0$ then $b = a$ and the length of the period of the continued fraction is 1, not 3.

i	P_i	Q_i	a_i
0	0	1	$b+1$
1	$b+1$	1	$2b+2$
2	$b+1$	1	$2b+2$

Table 1: PQa for $a = 1$

Before moving on to individual cases, we show that if $a = 1$ then $\lfloor\sqrt{D}\rfloor = b + 1$, while if $a > 1$ then $\lfloor\sqrt{D}\rfloor = b$. Clearly, $b^2 < D$ because $D = b^2 + 4(an + 1)$. Now,

$$(b+1)^2 - D = -4an + 2a + 2na^2 + 2n - 3 = -1 + 2(a-1) + 2n(a-1)^2$$

This is -1 if $a = 1$, and is positive if $a > 1$.

If $a = 1$, then $(b+2)^2 - D = 4(n+1) > 0$. So, $(b+1)^2 < D < (b+2)^2$, and $\lfloor\sqrt{D}\rfloor = b+1$. In fact, in this case, $b = 2n+1$, and $D = 4n^2+8n+5 = (2n+2)^2+1$.

If $a > 1$ then $b^2 < D < (b+1)^2$, so $\lfloor\sqrt{D}\rfloor = b$.

Now we turn to the individual cases. First suppose $a = 1$. We will show that $L_1 = 1$ in this case. Above we showed that when $a = 1$, $D = (2n+2)^2+1$, so D is a square plus 1. It is well known that the continued fraction expansion of $\sqrt{D}$ will have period 1 [2, p. 253, exercise 5.3.21], but for completeness that calculation is given in Table 1.

For the second case, suppose $a > 1$ is odd. Applying the PQa algorithm algebraically shows the length of the period of the continued fraction expansion of $\sqrt{D}$ is 5, as shown in Table 2. To check that this expansion is correct, one only has to check that

$$P_i = Q_{i-1}a_{i-1} - P_{i-1}$$

$$Q_i = (D - P_i^2)/Q_{i-1}, \text{ and}$$
$$0 \le \frac{P_i + b}{Q_i} - a_i < 1.$$

That this last test is valid follows from the fact that, since $b = \lfloor\sqrt{D}\rfloor$, P_i is an integer, and Q_i is a positive integer,

$$\left\lfloor \frac{P_i + b}{Q_i} \right\rfloor = \left\lfloor \frac{P_i + \sqrt{D}}{Q_i} \right\rfloor.$$

These equations are easily verified for each index i. Table 2 shows $z_i = \frac{P_i+b}{Q_i} - a_i$ written so as to make it easy to verify that $0 \leq z_i < 1$. Recall that $a \geq 3$.

For the third case, suppose $a = 2$. Then $L_1 = 11$ as shown in Table 3. Here everything can be written in terms of n; in particular, $b = 5n + 2$ and $D = 25n^2 + 28n + 8$.

Finally, for the fourth case, suppose $a > 2$ is even. Table 4 shows that $L_1 = 15$.

References

[1] Noburo Ishii, Pierre Kaplan, and Kenneth S. Williams, "On Eisenstein's problem," Acta Arithmetica, LIV (1990), pp 323-345.

[2] Richard A. Mollin, Fundamental Number Theory with Applications, CRC Press, Boca Raton, 1998.

[3] Kenneth S. Williams and Nicholas Buck, "Comparison of the Lengths of the Continued Fractions of sqrt(D) and (1+sqrt(D))/2," Proceedings of the AMS, v 120, no 4, April 1994, pp 995-1002.

i	P_i	Q_i	a_i	$\frac{P_i+b}{Q_i} - a_i$
0	0	1	b	0
1	b	$4(an+1)$	$(a-1)/2$	$\frac{1}{2} + \frac{1}{2(a+1/n)}$
2	$na^2 + a - 2an - n - 2$	$na^2 + a - n$	1	$1 - \frac{2[n(a-1)+1]}{a[n(a-1)+1]+n(a-1)}$
3	$2(an+1)$	$na^2 + a - n$	1	$\frac{2[n(a+1)+1]}{(a-1)[n(a+1)+1]+1}$
4	$na^2 + a - 2an - n - 2$	$4(an+1)$	$(a-1)/2$	0
5	b	1	$2b$	0
6	b	$4(an+1)$	$(a-1)/2$	$\frac{1}{2} + \frac{1}{2(a+1/n)}$

Table 2: PQa for $a > 1$, a odd

i	P_i	Q_i	a_i	$\frac{P_i+b}{Q_i} - a_i$
0	0	1	$5n+2$	0
1	$5n+2$	$8n+4$	1	$n/(4n+2)$
2	$3n+2$	$2n+1$	4	0
3	$5n+2$	4	$(5n+1)/2$	$1/2$
4	$5n$	$7n+2$	1	$(3n)/(7n+2)$
5	$2n+2$	$3n+2$	2	$n/(3n+2)$
6	$4n+2$	$3n+2$	2	$(3n)/(3n+2)$
7	$2n+2$	$7n+2$	1	$2/(7n+2)$
8	$5n$	4	$(5n+1)/2$	0
9	$5n+2$	$2n+1$	4	$(2n)/(2n+1)$
10	$3n+2$	$8n+4$	1	0
11	$5n+2$	1	$10n+4$	0
12	$5n+2$	$8n+4$	1	$n/(4n+2)$

Table 3: PQa for $a = 2$

i	P_i	Q_i	a_i	$\frac{P_i+b}{Q_i} - a_i$
0	0	1	b	0
1	b	$4(an+1)$	$a/2$	$\frac{1}{2(a+1/n)}$
2	$b-2n$	$an+1$	$2a$	0
3	b	4	$(b-1)/2$	$1/2$
4	$b-2$	$b+na$	1	$1-\frac{2(na+1)}{a(na+1)+n(a+1)}$
5	$na+2$	$b-na$	1	$\frac{2(na+1)}{(a-1)(na+1)+n+1}$
6	$b-2na-2$	$4(na+1)$	$a/2-1$	$\frac{1}{2}+\frac{1}{2(a+1/n)}$
7	$b-2na-2n-2$	$b-2n$	1	$1-\frac{2(1-1/a+1/(na))}{a-1/a+1/n}$
8	$2na+2$	$b-2n$	1	$\frac{2(1+1/a+1/(na))}{a-1/a+1/n}$
9	$b-2na-2n-2$	$4(na+1)$	$a/2-1$	$1/2$
10	$b-2na-2$	$b-na$	1	$1-\frac{2}{(a-1)(na)+n+a}$
11	$na+2$	$b+na$	1	$2/(b+na)$
12	$b-2$	4	$(b-1)/2$	0
13	b	$an+1$	$2a$	$\frac{2}{a+1/n}$
14	$b-2n$	$4(an+1)$	$a/2$	0
15	b	1	$2b$	0
16	b	$4(an+1)$	$a/2$	$\frac{1}{2(a+1/n)}$

Table 4: PQa for $a > 2$ even