THERE ARE NO 3×3 BINARY PALINDROMIC MAGIC SQUARES

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1. Introduction

Ed Pegg, Jr. reports that in 2001 David Clark asked whether there are any 3×3 magic squares so that every entry is a palindrome when written in base 2. See "material added 27 May 2001" at

www.mathpuzzle.com/Aug52001.htm,

and additional comments at

www.mathpuzzle.com/palmagicsquares.txt.

The main purpose of this note is to show that there are no such magic squares. A 3×3 *magic square* is a 3×3 array of 9 distinct positive integers so that each row, each column, and the two main diagonals add up to the same number, called the *magic constant*.

Before getting to the main proof, we give a few examples of arrays that almost meet the criteria and an example of a 4×4 magic square with every entry a binary palindrome. Following the proof of the main result, we show that arithmetic progressions of binary palindromes cannot have more than six terms.

2. Near misses and a 4×4 magic square

Here's a square where all entries are binary palindromes and the sums of the rows, columns, and one diagonal, but not the other, are all 477. In decimal:

195	27	255
231	153	93
51	297	129

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and in binary:

11000011	11011	11111111
11100111	10011001	1011101
110011	100101001	10000001

Here are three fully magic squares, where all but the third entry in the second row are binary palindromes (in decimal):

45	3	33
15	27	39
21	51	9

51	3	45
27	33	39
21	63	15

195	27	165
99	129	159
93	231	63

Here's a 4×4 magic square with every entry a binary palindrome. The magic constant is 188. In decimal:

3	1	119	65
7	127	21	33
85	9	31	63
93	51	17	27

and in binary:

11	1	1110111	1000001
111	1111111	10101	100001
1010101	1001	11111	111111
1011101	110011	10001	11011

It seems that there are 4×4 magic squares with every entry a binary palindrome with arbitrarily large magic constants.

Are there 5×5, or higher order, magic squares of binary palindromes?

3. General theory of 3×3 magic squares

It is well known that every 3×3 magic square of distinct integers can, up to rotation and reflection, be written as

$$\begin{array}{ccc} a+u+2v & a & a+2u+v \\ a+2u & a+u+v & a+2v \\ a+v & a+2u+2v & a+u \end{array}$$

with $0 < a$, $0 < u < v$, $v \neq 2u$, [1, p. 137] [2, p. 148] [3]. The magic constant is $3a + 3u + 3v$.

The terms above can be rearranged into another 3×3 array where every row is a three-term arithmetic progression, all with the same difference between terms; and every column is a three-term arithmetic progression, all with the same difference between terms, necessarily different from the row difference:

$$\begin{array}{ccc} a & a+u & a+2u \\ a+v & a+u+v & a+2u+v \\ a+2v & a+u+2v & a+2u+2v \end{array}$$

We call this an AP-array.

4. There are no 3×3 magic squares with every entry a binary palindrome

Define $D(a)$ as the number of digits of the positive integer a, written in base 2. Index these digits so that digit 0 is the rightmost digit, digit 1 is the next digit to the left, up to the leftmost digit, which has index $D(a) - 1$. Let a_i be the i-th digit of a. Let $I(a)$ be the smallest index i so that $a_i = 1$.

Lemma 1. *If a and $a + b$ are binary palindromes, $0 < b < a$, and $i = I(b)$ then*

$$D(a+b) = D(a) \iff a_i = 0, \text{ and}$$
$$D(a+b) > D(a) \iff D(a+b) = D(a) + 1 \iff a_i = 1.$$

Proof. Assume first that $D(a) = D(a + b)$. Let $n = D(a) = D(a + b)$. For $0 \leq j < i$, $(a + b)_j = a_j$. Because a and $a + b$ are palindromic, $(a+b)_j = a_j$ for $n-i-1 < j \leq n-1$. If $a_i = 1$ then $(a+b)_i = 0$. Then also $a_{n-i-1} = 1$ and $(a + b)_{n-i-1} = 0$. Because $a_{n-i-1} > (a + b)_{n-i-1}$ and $a_j = (a + b)_j$ for $j > n - i - 1$, we have that $a > a + b$. This contradiction shows that if $D(a+b) = D(a)$ then $a_i = 0$. (It is possible that $i = n - i - 1$.)

Now assume that $D(a) < D(a+b)$ and let $n = D(a)$. Because $a+b < 2a$ and $D(2a) = D(a)+1$, it follows that $D(a+b) = D(a)+1 = D(2a)$. Observe that $(a+b)_n = (2a)_n$ because the leading digit of each of these $n+1$-digit numbers is 1. Let $k > 0$ be the minimum integer so that $(a+b)_{n-k} \neq (2a)_{n-k}$. Then $(a+b)_{n-k} = 0$ and $(2a)_{n-k} = 1$, because $a+b < 2a$. But then, by palindromocity, $(a+b)_k = 0$ and $a_k = 1$. Also $(a+b)_j = a_j$ for $0 \leq j < k$. It follows that $b_j = 0$ for $0 \leq j < k$, and $b_k = 1$. Thus $i = k$ and so $a_i = 1$. (k can be as large as $n - 1$.) □

Lemma 2. *If a, $a+b$, $a+2b$ are binary palindromes, then $D(a+2b) > D(a)$.*

Proof. Each of a, $a+b$, and $a+2b$ must be odd, so b must be even. If $b > a$, then $a+b > 2a$, so $D(a+b) \geq D(2a) = D(a)+1$. As $D(a+2b) \geq D(a+b)$, $D(a+2b) > D(a)$. Now assume $b < a$, and let $i = I(b)$. If $a_i = 1$ then $D(a+b) > D(a)$, and so $D(a+2b) > D(a)$. Otherwise $(a+b)_i = 1$, and $D(a+2b) > D(a+b) = D(a)$. □

Theorem 1. *There are no 3×3 magic squares with every entry a binary palindrome.*

Proof. We will show there is no AP-array with every entry a binary palindrome. We use the notation from the section "General theory of 3×3 magic squares."

First suppose that $I(u) \neq I(v)$. Then $I(v+u) = I(v-u) = \min(I(u), I(v))$. Let $i = I(v+u) = I(v-u)$. Let $k = D(a+u+v)$. Observe that $v - u < v + u < a + u + v$.

If $(a+u+v)_i = 0$ then $D(a+2v) = D((a+u+v)+(v-u)) = k$ and $D(a+2u+2v) = D((a+u+v)+(v+u)) = k$ by Lemma 1. Then $a+2v$, $a+u+2v$, $a+2u+2v$ is a three-term arithmetic progression of binary palindromes, all with the same number of digits, which is impossible by Lemma 2.

If $(a+u+v)_i = 1$ then $D(a+2v) = D((a+u+v)+(v-u)) = k+1$ and $D(a+2u+2v) = D((a+u+v)+(v+u)) = k+1$. Again $a+2v$,

$a+u+2v$, $a+2u+2v$ is a three-term arithmetic progression of binary palindromes, all with the same number of digits, which is impossible.

Now suppose that $I(u) = I(v)$ and let $i = I(u) = I(v)$.

If $(a+u+v)_i = 1$, then

$$(a+v)_i = (a+2u+v)_i = (a+u+2v)_i = 0$$

and

$$(a+2v)_i = (a+2u+2v)_i = (a+2u)_i = 1.$$

Let $k = D(a+u+v)$. Then, by application of Lemma 1, $D(a+v) = k$, $D(a+2v) = k$, and $D(a+2u+v) = k+1$. It follows that $a+2v < a+2u+v$ and so $v < 2u$. Then $D(a+2u+v) = D((a+2u)+(v)) = D(a+2u)+1$, so $D(a+2u) = k$. But then $a+2u$, $a+u+v$, $a+2v$ is a three-term arithmetic progression of binary palindromes, all with the same number of digits, which is impossible.

If $(a+u+v)_i = 0$, then

$$(a+v)_i = (a+2u+v)_i = (a+u+2v)_i = 1$$

and

$$(a+2v)_i = (a+2u+2v)_i = (a+2u)_i = 0.$$

Let $k = D(a+v)$. Then $D(a+2v) = k+1$, and $D(a+u+v) = D(a+2u+v) = k+1$. In particular, $D(a+2v) = D(a+2u+v)$. Because $(a+2v)_j = (a+2u+v)_j$ for $0 \leq j < i$, it follows that $(a+2v)_j = (a+2u+v)_j$ for $k-i < j \leq k$. This says that the leading i digits of $a+2v$ and $a+2u+v$ are the same. But $(a+2v)_i = 0$ and $(a+2u+v)_i = 1$, so $(a+2v)_{k-i} = 0$ and $(a+2u+v)_{k-i} = 1$. This tells us that $a+2v < a+2u+v$, so $v < 2u$. Then $D(a+2u+v) = D(a+2u)$, so $D(a+2u) = k+1$. But then $a+2u$, $a+u+v$, $a+2v$ is a three-term arithmetic progression of binary palindromes, all with the same number of digits, which is impossible. □

5. Binary palindromes in arithmetic progression

The lemma immediately following shows that there cannot be arithmetic progressions of binary palindromes longer than 6 terms. APs of length 6 are possible: 3, 9, 15, 21, 27, 33, and 195, 585, 975, 1365, 1755, 2145 are examples.

Lemma 3. *There cannot be seven positive binary palindromes in arithmetic progression.*

Proof. Let a, $a+u$, $a+2u$, …, $a+5u$ be six binary palindromes. We will show that $a+6u$ cannot be a binary palindrome.

Let $i = I(u)$ and let k be the number of binary digits in $a+u$.

If $(a+u)_i = 1$ then $D(a+2u) = D(a+3u) = k+1$, $D(a+4u) = D(a+5u) = k+2$, and $D(a+6u) = k+3$. Because $D(a+6u) = k+3$ and $D(a+3u) = k+1$, it follows that $a+6u > 2(a+3u)$, which is clearly impossible.

If $(a+u)_i = 0$ then $D(a+2u) = k$, $D(a+3u) = D(a+4u) = k+1$, and $D(a+5u) = D(a+6u) = k+2$. Let $j = k-i$. Note that by palindromicity

$$(a+u)_j = (a+3u)_{j+1} = (a+5u)_{j+2} = 0$$

and

$$(a+2u)_j = (a+4u)_{j+1} = (a+6u)_{j+2} = 1.$$

Let Z be the number that is the rightmost i digits of $a+u$ (including leading zeros, if any), and let A be the number formed by reversing the digits of Z. Note that all of the numbers $a+u$, …, $a+6u$ end with the same i digits that form Z, and, by palindromicity, all begin with the digits of A. Note that $i \geq 1$, so $Z \geq 1$ and $A \geq 1$. Then,

$$a+u = A2^{j+1} + B$$

for some $B < 2^j$. Similarly,

$$a+2u = A2^{j+1} + 2^j + C$$

where $C \leq 2^j - 1$. Thus,

$$u = a+2u-(a+u) = A2^{j+1}+2^j+C-A2^{j+1}-B \leq 2^j+2^j-1 < 2^{j+1}.$$

so

$$u < 2^{j+1}.$$

Also

$$a+6u = A2^{j+3} + 2^{j+2} + E$$

where $E < 2^{j+2}$. So

$$4u = a+6u-(a+2u) \geq A2^{j+3} + 2^{j+2} - A2^{j+1} - 2^j - (2^j - 1)$$

$$= (3A + 1)2^{j+1} + 1 > 4 \cdot 2^{j+1}.$$

Because $u < 2^{j+1}$, this is impossible. □

References

[1] Martin Gardner, *Riddles of the Sphinx*, New Mathematical Library, MAA, 1987.

[2] Maurice Kraitchik, *Mathematical Recreations*, Dover, 1953.

[3] Problem E 3440, *American Mathematical Monthly*, 99 (1992), 966-967.