SEARCHING FOR 4×4 MAGIC SQUARES

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Abstract. We give a fairly detailed algorithm for doing a brute force search for 4×4 magic squares where every element has some desired property (e.g., squares, palindromes when written in binary).

1. Searching for 4×4 magic squares

We discuss methods to search for 4×4 magic squares with entries having given properties

A 4×4 *magic square* is a 4×4 array of 16 distinct positive integers so that every row, every column and each of the two main diagonals add to the same sum, called the *magic constant*. This note will give a fairly efficient way to search for 4×4 magic squares where each element has a certain special property, such as being a square or a palindrome when written in binary.

Create an array $v(*)$ with the first n integers with the given property (squares, binary palindromes, ...). You might want to precompute this array and store it somewhere, to be called up when you run the main routine.

We will use the following notation.

> nrc will the the entry for row r and column c, so $n11$ is the upper left entry, and $n24$ is the rightmost entry in the second row.
>
> irc is the index in $v(*)$ for nrc, so $nrc = v(irc)$.

Essentially what we will do for a given potential magic sum is loop on $n11$, $n12$, $n13$, $n41$, $n22$, $n23$, and $n21$. We will fill in as many other entries as possible at each stage. Note that $n11 + n14 + n41 + n44$ is the

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magic sum, a fact that might not be immediately obvious. If a deduced entry does not have the right form, we will exit to an appropriate loop.

As there are 32 squares that can be derived by rotation, reflection, and permutations of rows and columns of a given square, we will require that $n11 > n14 > n41$, $n11 > n44$, $n12 > n13$, and $n11$ is greater than any of $n22$, $n23$, $n32$, and $n33$.

With that background, here's the main algorithm.

Loop on magic sum. Note that for positive integers, the magic sum is at least 34, as this is the magic sum for a square with entries 1 to 16.

Loop on $i11$, get $n11 = v(i11)$. If $n11$ is less than the minimal possible value (at least one-fourth of the magic sum), go to the next $i11$. If $n11$ is greater than the magic sum minus the three smallest integers with the desired property, try the next largest magic sum.

Loop on $i12$, and get $n12 = v(i12)$. $i12$ can be anything 2 or larger, as long as $n12$ is less than the magic sum minus $n11$ and the two smallest number with the given property. If $i12 = i11$ then go to next $i12$.

Loop on $i13$, and get $n13 = v(i13)$. $i13$ can be anything 1 or larger, as long as $n12$ is less than the magic sum minus the quantity $n11$ plus $n12$ plus the smallest number with the given property. If $i13 = i11$ then go to next $i13$. If $i13 \geq i12$ then go to the next $i12$.

Deduce that $n14 = ms - n11 - n12 - n13$. If $n14 < 0.5$ then go to the next i12. If $n14 = n11$ or $n14 = n12$ or $n14 = n13$ then go to the next i13. If $n14$ do not have the desired property, then go to the next $i13$. If $n14 > n11$ Then go to the next $i13$.

Note that the tests for $n14$ are typical: each new nrc should be a positive integer, should not equal any previously set entry, and should have the desired property.

Loop on $i41$. $i41$ should not be any of $i11$, $i12$, or $i13$. If any of these occur, go to the next $i41$. If $n41 = v(41) \geq n14$, then go to the next $i13$. If $n41$ is greater than its maximum allowable value, go to the next $i13$.

Deduce $n44$ as $ms - n11 - n14 - n41$. If $n44 \leq 0$ then go to the next $i13$. If $n44$ equals any previously set number, $n44 > n11$, or $n44$ does not have the desired property, then go to the next $i41$.

Loop on $i22$. Note that $n22$ has to be less than $n11$, and that the maximum value of $n22$ is constrained by values already set in the second column, and in the main (upper left to lower right) diagonal. If $n22$ is too large by any of these tests, then go to the next $i41$. If $n22$ equals any previously set value, go to the next $i22$.

Set $n33 = ms - n11 - n22 - n44$. If $n33$ does not have the desired property, equals any previously set value, or is greater than $n11$, then go to the next $i22$. If $n33 \leq 0$ then go to the next $i41$.

Loop on $i23$. $n23$ has to be less than $n11$ and the maximum value of $n23$ is constrained by values in the row, column, and diagonal already set; if $n23$ exceeds any of these constraints, go to the next $i22$. If $n23$ equals any previously set value, go to the next $i23$.

Set $n43 = ms - n13 - n23 - n33$ and $n32 = msum - n14 - n23 - n41$. If either does not have the desired property, or either equals a previously set value, or $n32 \geq n11$, go to the next $i23$. If either is zero or negative, go to the next $i22$.

Set $n42 = ms - n41 - n43 - n44$. If $n42$ does not have the desired property, equals any previously set value, or is less than 1, go to the next $i23$.

Loop on $i21$; this is the last loop. The maximum value of $n21$ is constrained by values set in the same column and row. If $n21$ exceeds this maximum, go to the next $i23$. If $n21$ equals any previously set value, go to the next $i21$.

Set $n31 = ms - n11 - n21 - n41$, $n24 = ms - n21 - n22 - n23$, and $n34 = ms - n31 - n32 - n33$. If $n31 < 1$ or $n24 < 1$ then go to the next $i23$. If any are equal to a previously set value, or don't have the desired property, or $n34 < 1$ the go to the next $i21$.

If you are here, then you have a completely magic square where every entry has the desired property. You should print it out.

Do next $i21$, $i23$, $i22$, $i41$, $i13$, $i12$, $i11$.

Sometimes, additional tests can speed up the search. For example, if searching for 4×4 magic squares of squares, some tests of whether certain sums of remaining values can be sums of 2 or 3 squares can speed up the search.

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