Nearest Square Continued Fractions

and Related Topics

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New Results

- For any quadratic irrational the lengths of the periods of the nearest square continued fraction (NSCF) and the nearest integer continued fraction (NICF) are the same
- The mid-point of the period of the NSCF of $\sqrt{D}$ can be recognized using three tests
- More direct proof of the formulas that recognize a quadratic irrational with a purely periodic NICF expansion

Half-Regular Continued Fractions

$$\xi_0 = a_0 + \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + \cfrac{\varepsilon_3}{a_3 + \cdots}}} = a_0 + \frac{\varepsilon_1 |}{| a_1} + \frac{\varepsilon_2 |}{| a_2} + \cdots$$

$\varepsilon_i = \pm 1, \quad a_i \geq 1$ for $i \geq 1$

$a_i + \varepsilon_{i+1} \geq 1$ for $i \geq 1$

Main Continued Fraction Step

For $\xi_i = \dfrac{P_i + \sqrt{D}}{Q_i}$,

$$\xi_{i+1} = \frac{P_{i+1} + \sqrt{D}}{Q_{i+1}} = \frac{\varepsilon_{i+1}}{\xi_i - a_i} > 1$$

Basic Continued Fraction Steps

$$\xi_i = \frac{P_i + \sqrt{D}}{Q_i} = c + \frac{Q'_{i+1}}{P'_{i+1} + \sqrt{D}} = c + 1 - \frac{Q''_{i+1}}{P''_{i+1} + \sqrt{D}}$$

$$c = \lfloor \xi_i \rfloor = \left\lfloor \left(P_i + \sqrt{D}\right) / Q_i \right\rfloor$$

$$P'_{i+1} = cQ_i - P_i, \quad Q'_{i+1} = \left(D - {P'_{i+1}}^2\right) / Q_i$$

$$P''_{i+1} = (c+1)Q_i - P_i, \quad Q''_{i+1} = \left({P''_{i+1}}^2 - D\right) / Q_i$$

$$P''_{i+1} = P'_{i+1} + Q_i, \quad Q''_{i+1} = P''_{i+1} + P'_{i+1} - Q'_{i+1}$$

Definitions of Continued Fractions

- For RCF, $a_i = c = \lfloor \xi_i \rfloor$ and $\varepsilon_{i+1} = +1$
- For NICF
  - o $a_i = \lfloor \xi_i \rceil$, the nearest integer to ξ_i
  - o $\varepsilon_{i+1} = +1$ if $a_i = c$, $\varepsilon_{i+1} = -1$ if $a_i = c + 1$
- For NSCF
  - o If $Q_i > 0$ and $|Q''_{i+1}| > |Q'_{i+1}|$ or $Q_i < 0$ and $|Q''_{i+1}| \geq |Q'_{i+1}|$ then $a_i = c$ and $\varepsilon_{i+1} = +1$
  - o Otherwise $a_i = c + 1$ and $\varepsilon_{i+1} = -1$

Next Up:

L-NICF = L-NSCF

What Drives Differences Among RCF, NICF, and NSCF?

- How RCF, NICF, and NSCF differ is driven largely by the strings of m consecutive RCF partial quotients that are 1, called $m-$*unisequences*
- If no RCF partial quotients are 1, then RCF, NICF, and NSCF are all the same
- If there are RCF partial quotients of 1, NICF and NSCF tend to skip every other RCF step

A Selenius Lemma

Selenius [6, p. 62]—Let ξ_i be a complete quotient for the RCF of $\sqrt{D}$, with positive and negative representations for $i \geq 0$

$$\xi_i = \frac{P_i + \sqrt{D}}{Q_i} = a_i + \frac{Q_{i+1}}{P_{i+1} + \sqrt{D}} = a_i + 1 - \frac{Q''_{i+1}}{P''_{i+1} + \sqrt{D}}$$

where $a_i = \lfloor \xi_i \rfloor$. Then

(1) If $a_{i+1} = 1$ then $\dfrac{P''_{i+1} + \sqrt{D}}{Q''_{i+1}} = \xi_{i+2} + 1$

(2) If $a_{i+1} \geq 2$, then $Q_{i+1}'' - Q_{i+1} > 0$ (thus $Q_{i+1}'' \leq Q_{i+1}$ implies that $a_{i+1} = 1$)

RCF versus NSCF for (13 + √257)/11

Regular CF						Nearest Square CF				
Index	P_i	Q_i	a_i	Q_i''		Index	$\tilde{P}_i$	$\tilde{Q}_i$	$\tilde{a}_i$	$\tilde{\varepsilon}_i$
0	13	11	2			0	13	11	3	1
1	9	16	1	13						
2	7	13	1	17		1	20	13	3	-1
3	6	17	1	8						
4	11	8	3	31		2	19	8	4	-1
5	13	11	2	23		3	13	11	3	1
6	9	16	1	13						

The Q-γ Law of Selenius

Let $\theta_n = B_n \, |B_n \xi_0 - A_n|$, where A_n/B_n is the n-th RCF convergent to ξ_0. Suppose Q_n, B_n are positive for all $n \geq 0$.

(a) If n is sufficiently large (e.g., $B_n B_{n-1} \geq Q_0$) and $Q_{n+1} \neq Q_n$, then (*) $Q_{n+1} < Q_n \Leftrightarrow \theta_n < \theta_{n-1}.$

Moreover if $\xi_0 = \sqrt{D}$, then equation (*) holds for $n \geq 1$.

(b) If $Q_{n+1} = Q_n$, and $n \geq 1$, then

$$(-1)^n \, (\theta_n - \theta_{n-1}) > 0.$$

Selenius (Satz 29 [6, p. 52]) stated his result in terms of $\gamma_n = 1/\theta_{n-1}$

4-Unisequences
NICF and NSCF for $(21 + \sqrt{675})/18$

Regular CF					Nearest Integer CF						Nearest Square CF			
i	P_i	Q_i	a_i		I	$\tilde{P}_i$	$\tilde{Q}_i$	$\tilde{a}_i$	$\tilde{\varepsilon}_i$		$\tilde{P}_i$	$\tilde{Q}_i$	$\tilde{a}_i$	$\tilde{\varepsilon}_i$
0	21	18	2		0	21	18	3	1		21	18	3	1
1	15	25	1											
2	10	23	1		1	33	23	3	-1		33	23	2	-1
3	13	22	1								13	22	2	1
4	9	27	1		2	36	27	2	-1					
5	18	13	3		3	18	13	3	1		31	13	4	-1
6	21	18	2		4	21	18	3	1		21	18	3	1

6-Unisequences
NSCF for $(P_0 + \sqrt{4623})/Q_0$

	NSCF of $(55 + \sqrt{4623})/47$					NSCF of $(55 + \sqrt{4623})/34$			
i	$\tilde{P}_i$	$\tilde{Q}_i$	$\tilde{a}_i$	$\tilde{\varepsilon}_i$		$\tilde{P}_i$	$\tilde{Q}_i$	$\tilde{a}_i$	$\tilde{\varepsilon}_i$
0	55	47	3	1		55	34	4	1
1	86	59	2	-1		81	57	3	-1
2	32	61	2	1		90	61	2	-1
3	90	57	3	-1		32	59	2	1
4	81	34	4	-1		86	47	3	-1
5	55	47	3	1		55	34	4	1
RCF = <2, 1, 1, 1, 1, 1, 1, 3>						RCF = <3, 1, 1, 1, 1, 1, 1, 2>			

NSCFs for 2−Unisequences

	NSCF of (8 + √99)/5					NSCF of (8 + √99)/7			
i	$\tilde{P}_i$	$\tilde{Q}_i$	$\tilde{a}_i$	$\tilde{\varepsilon}_i$		$\tilde{P}_i$	$\tilde{Q}_i$	$\tilde{a}_i$	$\tilde{\varepsilon}_i$
0	8	5	4	1		8	7	2	1
1	12	9	2	-1		6	9	2	1
2	6	7	2	1		12	5	4	-1
3	8	5	4	1		8	7	2	1
	RCF = <3, 1, 1, 2>					RCF = <2, 1, 1, 3>			

Summary

- In an *m*-unisequence the NICF always has jumps of 2, and the NSCF mostly does
- If there is an *m*-unisequence
  - of odd length, NSCF has only jumps of 2
  - of length divisible by 4, the NSCF has a jump of 1 right in the middle
  - of length congruent to 2 modulo 4, NSCF has a jump of 1 just before or just after the middle

A Little Notation

- L-RCF, L-NICF, L-NSCF are the lengths of the periods of the continued fractions
- N-NICF, N-NSCF are the number of $\varepsilon_i = -1$ in the period of the continued fraction

Proof of Equality of Lengths of Periods

- By following the *m*-unisequence jumps, we show that
  - L-NICF + N-NICF = L-RCF
    - Each jump of 2 gives one $\varepsilon_{i+1} = -1$
  - L-NSCF + N-NSCF = L-RCF
    - Same reason
  - N-NSCF = N-NICF
    - Each *m*-unisequence gives $\lfloor (m+1)/2 \rfloor$ cases of $\varepsilon_{i+1} = -1$
- These imply that L-NICF = L-NSCF

Corollaries

- 0.500 L-RCF $\leq$ L-NSCF $\leq$ L-RCF
  - o Follows from result for NICF
- L-NSCF $\rightarrow$ 0.694 L-RCF in two senses
  - o True ``almost everywhere''; conjectured for quadratic surds
  - o The 0.694 above is actually $\log(\phi)/\log(2)$ = 0.6942419…, where $\phi = (1 + \sqrt{5})/2$

Next Up:

Midpoint Criteria for NSCF of $\sqrt{D}$

Classical Mid-Point Criteria For RCF and NICF of √D

- Mid-point criteria for RCF of √*D*:
  - If $P_i = P_{i+1}$ then $\ell = 2i$
  - If $Q_i = Q_{i+1}$ then $\ell = 2i + 1$
- Williams and Buhr [7] give 6 mid-point criteria for the NICF of √*D* (which reduce to 5 with the sign convention we use here)

Symmetry in NSCF of $\sqrt{D}$

Type I—No complete quotient of the cycle has the form $\left(p+q+\sqrt{p^2+q^2}\right)/p$ with $p > 2q > 0$, gcd(p, q) = 1. This type possesses the classical symmetries of the regular continued fraction for period length $k > 1$:

$$a_i = a_{k-i} \qquad 1 \le i \le k-1$$

$$Q_i = Q_{k-i} \qquad 1 \le i \le k-1$$

$$\varepsilon_i = \varepsilon_{k+1-i} \qquad 1 \le i \le k$$

$$P_i = P_{k+1-i} \qquad 1 \le i \le k$$

Example with $k = 5$: $\sqrt{91} = 10 - \frac{1|}{|2} + \frac{1|}{|6} - \frac{1|}{|6} + \frac{1|}{|2} - \frac{1|}{|20} - \frac{1|}{|2} \ldots$

Symmetry in NSCF of √D

Type II—There is one complete quotient ξ_i in the cycle of the form $\left(p+q+\sqrt{p^2+q^2}\right)/\,p$ where $p > 2q > 0$. In this case $k \geq 4$ is even and $i = k/2$. This type also possesses the symmetries of Type I, apart from

$$a_{k/2} = 2,\ \varepsilon_{k/2} = -1,\ \varepsilon_{k/2+1} = 1,$$

$$a_{k/2+1} = a_{k/2-1} - 1,\ P_{k/2} \neq P_{k/2+1}$$

Example with $k = 6$:

$$\sqrt{97} = 10 - \frac{1|}{|7} - \frac{1|}{|3} - \frac{1|}{|2} + \frac{1|}{|2} - \frac{1|}{|7} - \frac{1|}{|20} - \frac{1|}{|7} \dots$$

Mid-Point Criteria for NSCF of √*D*

- Three tests:
  - For Type I—*P*-test and *Q*-test
  - For Type II—*PQ*-test

- *P*-test: If $\widetilde{P}_i = \widetilde{P}_{i+1}$ then $\ell = 2i$

$$A_{\ell-1} = A_i B_{i-1} + \varepsilon_i A_{i-1} B_{i-2}, \quad B_{\ell-1} = B_{i-1}\left(B_i + \varepsilon_i B_{i-2}\right)$$

$$A_{\ell-1}^2 - DB_{\ell-1}^2 = 1$$

Mid-Point Criteria for NSCF of √*D*

- *Q-test:* If $\tilde{Q}_i = \tilde{Q}_{i+1}$ then $\ell = 2i + 1$

$$A_{\ell-1} = A_i B_i + \varepsilon_{i+1} A_{i-1} B_{i-1}, \quad B_{\ell-1} = B_i^2 + \varepsilon_{i+1} B_{i-1}^2$$

$$A_{\ell-1}^2 - D B_{\ell-1}^2 = -\varepsilon_{i+1}$$

- *PQ*-test: If $\tilde{P}_i = \tilde{Q}_i + \frac{1}{2}\tilde{Q}_{i-1}$ and $\varepsilon_i = -1$ then $\ell = 2i$

$$A_{\ell-1} = A_i B_{i-1} - B_{i-2}\left(A_{i-1} - A_{i-2}\right),$$

$$B_{\ell-1} = 2B_{i-1}^2 - B_i B_{i-2}$$

$$A_{\ell-1}^2 - D B_{\ell-1}^2 = -1$$

NSCF of √137
PQ-test: 15 = 11+8/2 and ε_3 = -1

Index	P_i	Q_i	a_i	ε_i		A_i	B_i
0	0	1	12	1		12	1
1	12	7	3	-1		35	3
2	9	8	3	1		117	10
3	15	11	2	-1		199	17
4	7	8	2	1		515	44
5	9	7	3	1		1744	149
6	12	1	24	-1		41341	3532
7	12	7	3	-1		122279	10447

Next Up:

NICF-Reduced Quadratic Irrationals

Reduced NICF QI's

- The NICF for a quadratic irrational ξ is purely periodic if and only if $\xi > 2$ and the conjugate of ξ satisfies

$$-0.618 = \frac{1-\sqrt{5}}{2} < \bar{\xi} \le \frac{3-\sqrt{5}}{2} = 0.382$$

- Hurwitz[4] proves this; our proof is more direct, self-contained, and makes minimal use of Hurwitz' singular continued fractions

NICF of $(312 + \sqrt{675})/207$

Index	P_i	Q_i	a_i	ε_i		ξ_i	ξ_i-bar	NICF-Reduced?
0	312	207	2	1		1.633	1.382	No
1	102	47	3	-1		2.723	1.617	No
2	39	18	4	-1		3.610	0.723	No
3	33	23	3	-1		2.564	0.305	Yes
4	36	27	2	-1		2.296	0.371	Yes
5	18	13	3	1		3.383	-0.614	Yes
6	21	18	3	1		2.610	-0.277	Yes
7	33	23	3	-1		2.564	0.305	Yes

Questions?

References

[1] A.A.K. Ayyangar, Theory of the nearest square continued fraction, *J. Mysore Univ.* Sect. A. 1(1941) 97-117.

[2] A.A.K. Ayyangar, New light on Bhaskaras Chakravala or cyclic method of solving indeterminate equations of the second degree in two variables, *J. Indian Math. Soc.*, 18, 1929--30, pp. 225--248.

[3] A.A.K. Ayyangar, A new continued fraction, *Current Sci.* 6, June 1938.

The Ayyangar papers are available at http://www.ms.uky.edu/~sohum/AAK/PRELUDE.htm

References

[4] A. Hurwitz, Uber eine besondere Art der Kettenbruch-Entwicklung reeller Grössen, *Acta Mathematica*, vol. 12, pp. 367-405, 1889.

[5] Keith R. Matthews, John P. Robertson, Jim White, Midpoint criteria for solving Pell's equation using the nearest square continued fraction, (submitted), available at http://www.numbertheory.org/pdfs/nscf_midpoint.pdf

[6] C.-O. Selenius, Konstruction und Theorie Halbregelmässiger Kettenbrüche mit idealer relativer Approximation, *Acta Acad. Abo. Math. Phys.* 22(1960), 3-77.

References

[7] H. C. Williams and P. A. Buhr, Calculation of the Regulator of $\mathbf{Q}(\sqrt{D})$ by use of the Nearest Integer Continued Fraction Algorithm, *Mathematics of Computation*, Vol. 33, No. 145. (Jan., 1979), pp. 369-381, corrigenda January 2009.

Appendix

Midpoint Criteria for NICF

1. $P_{i+1} = P_i, \quad \ell = 2i$
2. $P_{i+1} = P_i + Q_i, \quad \ell = 2i$
3. $Q_{i+1} = Q_i$ and $\varepsilon_{i+1} = -1, \quad \ell = 2i + 1$
4. $Q_{i+1} = Q_i$ and $\varepsilon_{i+1} = +1, \quad \ell = 2i + 1$
5. $P_{i+1} = Q_i + (Q_{i+1})/2, \quad \ell = 2i + 1$
6. $P_{i+1} = (Q_i)/2 + Q_{i+1}, \quad \ell = 2(i + 1)$

Comparison of Tests

Type of CF	Test					
NICF	1	2	3	4	5	6
NSCF	P	P	Q	Q	Q	PQ
RCF	P	P	P	Q	Q	Q
Count	19.8	2.6	0.3	1.3	0.3	0.7

Count is number of occurrences for $\sqrt{D}$, D not a square, up to 25 million, in millions

Comparison of NSCF and RCF periods for $\sqrt{D}$.

n	$\Pi(n)$	$P(n)$	$\Pi(n)/P(n)$
1,000,000	152,198,657	219,245,100	0.6941941
2,000,000	417,839,927	601,858,071	0.6942499
3,000,000	755,029,499	1,087,529,823	0.6942609
4,000,000	1,149,044,240	1,655,081,352	0.6942524
5,000,000	1,592,110,649	2,293,328,944	0.6942356
6,000,000	2,078,609,220	2,994,112,273	0.6942322
7,000,000	2,604,125,007	3,751,067,951	0.6942356
8,000,000	3,165,696,279	4,559,939,520	0.6942408
9,000,000	3,760,639,205	5,416,886,128	0.6942437
10,000,000	4,387,213,325	6,319,390,242	0.6942463

$\Pi(n)$ is the sum of the NSCF period lengths of $\sqrt{D}$ up to n, D not a square
$P(n)$ is the same for RCF
$\log((1 + \sqrt{5})/2)/\log(2) = 0.6942419136 \ldots$

Reduced QI for RCF and NSCF

- A quadratic irrational ξ_0 is RCF-reduced if
  1. $\xi_0 > 1$
  2. $-1 < \overline{\xi}_0 < 0$
- For NSCF
  - A quadratic irrational ξ_0 is ``special'' if
    $Q_0^2 + \frac{1}{4} Q_1^2 \leq D$, and $Q_1^2 + \frac{1}{4} Q_0^2 \leq D$
  - The successor of a special surd is ``semi-reduced''
  - The successor of a semi-reduced surd is ``reduced''
- A reduced surd always has a purely periodic continued fraction expansion; special and semi-reduced surds do not necessarily have purely periodic continued fraction expansions

One Step of NICF or NSCF Can be Two Steps of RCF

- Let x = fractional part of ξ_i, assume that $0.5 < x < 1.0$. Then:
- For RCF
  - o $1 < \xi_{i+1} = 1/x < 2$,
  - o $a_{i+1} = 1$, and
  - o $\xi_{i+2} = 1/(\xi_{i+1} - a_{i+1}) = 1/(1/x - 1) = x/(1 - x)$
- For NICF, $\tilde{\xi}_{i+1} = 1/(1 - x)$
- But $1/(1 - x) - x/(1 - x) = 1$