Most of this is Keith Matthews’ work, but Jim and I had a hand in it.
New Results

- For any quadratic irrational the lengths of the periods of the nearest square continued fraction (NSCF) and the nearest integer continued fraction (NICF) are the same.
- The mid-point of the period of the NSCF of $\sqrt{D}$ can be recognized using three tests.
- More direct proof of the formulas that recognize a quadratic irrational with a purely periodic NICF expansion.

We’ll define the NSCF and NICF shortly.
Mostly we’ll talk about nearest square continued fractions.
This is a variant of the very familiar regular or simple continued fraction, and a variant of the nearest integer continued fraction.
The NSCF is derived from the cyclic method of Bhaskara, as developed by A. A. Krishnaswami Ayyangar [1, 2, 3].

Last result not new; we have a simpler proof.
This is to show notation conventions we'll use:

Other sign conventions are possible:

For NICF and NSCF, epsilon_i and a_i satisfy additional relations:

We'll put tildes over variables for NICF and NSCF when we need to distinguish them from the RCF.
Main Continued Fraction Step

For $ξ_i = \frac{P_i + \sqrt{D}}{Q_i}$,

$ξ_{i+1} = \frac{P_{i+1} + \sqrt{D}}{Q_{i+1}} = \frac{ε_{i+1}}{ξ_i - a_i} > 1$

We use half-regular cfs: \(\xi_{i+1} > 1\); sign of \(e_{i+1}\) picked accordingly

\(a_i\) will always be either greatest integer less that \(\xi_i\) or least integer greater than \(\xi_i\);
\(e_{i+1}\) is 1 or -1 accordingly
Basic Continued Fraction Steps

\[\xi_i = \frac{P_i + \sqrt{D}}{Q_i} = c + \frac{Q'_{i+1}}{P'_{i+1} + \sqrt{D}} = c + 1 - \frac{Q''_{i+1}}{P''_{i+1} + \sqrt{D}}\]

\[c = \lceil \xi_i \rceil = \left\lfloor \left(\frac{P_i + \sqrt{D}}{Q_i}\right) \right\rfloor\]

\[P'_{i+1} = cQ_i - P_i, \quad Q'_{i+1} = \left(D - P'^2_{i+1}\right)/Q_i\]

\[P''_{i+1} = (c+1)Q_i - P_i, \quad Q''_{i+1} = \left(P''^2_{i+1} - D\right)/Q_i\]

\[P''_{i+1} = P'_{i+1} + Q_i, \quad Q''_{i+1} = P''_{i+1} + P'_{i+1} - Q'_{i+1}\]

For each cf—RCF, NICF, NSCF—a_i will be c or c+1, and \(\xi_{i+1}\) will be either the primed or double-primed (\(P + \sqrt{D})/Q\).

This gives formulas for \(\xi_{i+1}\) when a_i is floor or ceiling of \(\xi_i\).

\(c+1\) is the ceiling of \(\xi_i\).

Last line is computationally more efficient; gets \(P'', Q''\) without any additional multiplications or divisions, just additions and subtraction.
Start with $\xi_0$ in all cases

For all, epsilon_0 = +1

All are eventually periodic for any quadratic irrational

NSCF is “nearest square” because $P^2$ is the nearest to $D$ (see previous slide)
Next Up:

L-NICF = L-NSCF

Now we give a short summary of the ideas behind the proof that the lengths of the periods of the NICF and NSCF are the same.
What Drives Differences Among RCF, NICF, and NSCF?

- How RCF, NICF, and NSCF differ is driven largely by the strings of \( m \) consecutive RCF partial quotients that are 1, called \( m \)-unisequences.

- If no RCF partial quotients are 1, then RCF, NICF, and NSCF are all the same.

- If there are RCF partial quotients of 1, NICF and NSCF tend to skip every other RCF step.

Strings of consecutive partial quotients that are 1 in the RCF are what drives the differences among RCF, NICF, and NSCF.

When RCF partial quotients are greater than 1, NICF and NSCF are essentially the same, in a way we’ll make precise.

The main idea is that in an \( m \)-unisequence, NICF and NSCF tend to skip every other RCF step, with some exceptions for the NSCF.
A Selenius Lemma

Selenius [6, p. 62]—Let $\xi_{i}$ be a complete quotient for the RCF of $\sqrt{D}$, with positive and negative representations for $i \geq 0$

$$\xi_{i} = \frac{P_{i} + \sqrt{D}}{Q_{i}} = a_{i} + \frac{Q_{i+1}}{P_{i+1} + \sqrt{D}} = a_{i} + 1 - \frac{Q_{i+1}''}{P_{i+1}'' + \sqrt{D}}$$

where $a_{i} = \lfloor \xi_{i} \rfloor$. Then

(1) If $a_{i+1} = 1$ then $\frac{P_{i+1}'' + \sqrt{D}}{Q_{i+1}''} = \xi_{i+2} + 1$

(2) If $a_{i+1} \geq 2$, then $Q_{i+1}'' - Q_{i+1} > 0$ (thus $Q_{i+1}'' \leq Q_{i+1}$ implies that $a_{i+1} = 1$)

(1) When $a_{i} = 1$, then if we pick the partial quotient as the integer larger than $\xi_{i-1}$, then one step for that CF is essentially 2 RCF steps, because we get $\xi_{i+1} + 1$ as the next

(2) When $a_{i}$ is at least 2, then the NSCF (and NICF) will do one RCF step
This illustrates how NSCF and NICF skip every other RCF step when there’s an m-unisequence; NICF = NSCF for \((13 + \sqrt{257})/11\)

Has 3-uniseq

Note that in m-uniseq, \(Q''_i = Q_{i+1}\)

\(Q''_1 = 13 < Q_1 = 16\) so NSCF opts for \(Q''_1\), which is \(Q_2\). Then \(Q''_3 = 8 < Q_3 = 17\), so NSCF opts for \(Q''_3\) which is \(Q_4\). Then \(Q''_5 = 23 > Q_5 = 11\), so NSCF goes for \(Q_5\).

Note that while the NICF singularization works for an arbitrary quadratic surd, the NSCF singularization requires some conditions such as $\xi_0 \neq \sqrt{D}$, or $\xi_0$ is RCF reduced or $0 < Q_0 < 2 \sqrt{D}$ and $\xi_1$ is RCF reduced.
The \( Q - \gamma \) Law of Selenius

Let \( \theta_n = B_n |B_n - A_n| \), where \( A_n/B_n \) is the \( n \)-th RCF convergent to \( \zeta_0 \). Suppose \( Q_n, B_n \) are positive for all \( n \geq 0 \).

(a) If \( n \) is sufficiently large (e.g., \( B_n B_{n-1} \geq Q_0 \)) and \( Q_{n+1} \neq Q_n \), then \( (*) \ Q_{n+1} < Q_n \Leftrightarrow \theta_n < \theta_{n-1} \).

Moreover if \( \xi_0 = \sqrt{D} \), then equation (*) holds for \( n \geq 1 \).

(b) If \( Q_{n+1} = Q_n \) and \( n > 1 \), then

\[
(-1)^n (\theta_n - \theta_{n-1}) > 0.
\]

Selenius (Satz 29 [6, p. 52]) stated his result in terms of \( \gamma_n = 1/\theta_{n-1} \)

The theta’s tell us how well the convergents approximate \( \xi_0 \) and inform on the singularization process.

For example, if \( \text{theta}_n > \text{theta}_{(n+1)} \) then \( Q_{(n+1)} > Q_{(n+2)} \) and so \( Q_{(n+1)} > Q''_{(n+1)} \) and we get a jump of 2

\[ Q_{(n+2)} = Q''_{(n+1)} \text{ when } a_{(i+1)} = 1 \]
For an m-unisequence with even m, NSCF does a jump of 1 somewhere in the middle.

If 4 divides m, then the jump of 1 always occurs exactly in the middle.

At each RCF step, in the 4-uniseq, Q' and Q'' are the next two Qs. So at i = 0, Q'_1 = 25, Q''_1 = 23, so NSCF opts for the 23; not shown, theta_7 = .443 < theta_6 = .481, so Q_2 < Q_1, so Q''_1 < Q_1. (B’s not big enough for theorem to apply for theta_1 and theta_0). (n=0, k=4, t=2)

At i = 2, Q'_3 = 22, Q''_3 = 27, so NSCF opts for the 22; theta_2 = 0.440 < theta_3 = .510—says Q_3 < Q_4.

At i = 3, Q'_4 = 27, Q''_4 = 13, so NSCF opts for the 13; theta_4 = 0.251 < theta_3 = 0.510.
When $m$ is $2 \mod 4$, the jump of 1 in the middle occurs in one of two different places.

Both of these quadratic irrationals have 6-uniseqs, as shown at the bottom. $\epsilon_i = -1$ signifies a jump of 2; $\epsilon_i = +1$ signifies a jump of 1.

RCF has period length 8. $B_2 B_3 = 6 < 34 < 47$. For $Q_0 = 47$, $\theta_{10} = 0.449 < \theta_{11} = 0.456$, so $Q_3 < Q_4$; for $Q_0 = 34$, $\theta_{10} = 0.456 > \theta_{11} = 0.449$ so $Q_3 > Q_4$ (n=0, m=6, k=3, t=1)

### 6-Unisequences

**NSCF for $(P_0 + \sqrt{4623})/Q_0$**

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<th>$\bar{q}_i$</th>
<th>$\bar{q}_i$</th>
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RCF = <2, 1, 1, 1, 1, 1, 1, 3>  
RCF = <3, 1, 1, 1, 1, 1, 2>
NSCFs for 2–Unisequences

<table>
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<tr>
<th></th>
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<th>NSCF of $(8 + \sqrt{99})/7$</th>
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<tr>
<td>3</td>
<td>8</td>
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</table>

RCF = $<3, 1, 1, 2>$

RCF = $<2, 1, 1, 3>$

Even happens for 2-unisequences

$(8+\sqrt{99})/5$ have the same NICF and NSCF; RCF period is length 4; $\theta_5 = 0.251 < \theta_6 = 0.452$

$(8+\sqrt{99})/7$ have different NICF and NSCF; $\theta_5 = 0.502 > \theta_6 = 0.251$
Summary

- In an $m$ uniscquence the NICF always has jumps of 2, and the NSCF mostly does

- If there is an $m$-unisequence
  - of odd length, NSCF has only jumps of 2
  - of length divisible by 4, the NSCF has a jump of 1 right in the middle
  - of length congruent to 2 modulo 4, NSCF has a jump of 1 just before or just after the middle

We will see that however the jumps of 2 occur, the NICF and NSCF always have the same number of jumps of 2
A Little Notation

• L-RCF, L-NICF, L-NSCF are the lengths of the periods of the continued fractions
• N-NICF, N-NSCF are the number of \( \varepsilon_i = -1 \) in the period of the continued fraction

N-NICF and N-NSCF are also the number of jumps of 2
Proof of Equality of Lengths of Periods

- By following the $m$-unisequence jumps, we show that
  - $L\text{-NICF} + N\text{-NICF} = L\text{-RCF}$
    - Each jump of 2 gives one $\epsilon_{i+1} = -1$
  - $L\text{-NSCF} + N\text{-NSCF} = L\text{-RCF}$
    - Same reason
  - $N\text{-NSCF} = N\text{-NICF}$
    - Each $m$-unisequence gives $\lfloor(m+1)/2\rfloor$ cases of $\epsilon_{i+1} = -1$
- These imply that $L\text{-NICF} = L\text{-NSCF}$

Each jump of 2 makes period of $L\text{-NICF}$ 1 shorter than period of $L\text{-RCF}$, and adds 1 to $N\text{-NICF}$, so $L\text{-NICF} + N\text{-NICF} = L\text{-RCF}$

and, the jumps of 2 end at the right places

For NICF the count of $\epsilon_{i} = -1$ is easy; for NSCF depends on proving what we showed on the second previous slide; in turn this follows from the approximations given by the Q-\gamma law of Selenius
Corollaries

- 0.500 \text{L-RCF} < \text{L-NSCF} < \text{L-RCF}
  - Follows from result for NICF

- L-NSCF $\rightarrow$ 0.694 L-RCF in two senses
  - True ``almost everywhere''; conjectured for quadratic surds
  - The 0.694 above is actually $\log(\phi)/\log(2) = 0.6942419...$, where $\phi = (1 + \sqrt{5})/2$

L-NSCF is between half of L-RCF and RCF

There are two senses in which L-NSCF $\rightarrow$ 0.7 L-RCF:
  a) (sum L-NSCF)/(sum L-RCF) goes to 0.7
  b) as D gets larger, the probability that $|\text{L-NSCF}(\sqrt{D})/\text{L-RCF}(\sqrt{D}) - 0.7| < \epsilon$ goes to 1 for any epsilon

The result "0.69..RCF" is an almost everywhere result, and as you remark, was proved by W. Webb. It remains an experimentally observed, but unproved, phenomenon for quadratic surds.
We can recognize the midpoint of the NSCF of $\sqrt{D}$ without computing the whole period
Classical Mid-Point Criteria
For RCF and NICF of $\sqrt{D}$

- Mid point criteria for RCF of $\sqrt{D}$:
  - If $P_i = P_{i+1}$ then $\ell = 2i$
  - If $Q_i = Q_{i+1}$ then $\ell = 2i + 1$

- Williams and Buhr [7] give 6 mid-point criteria for the NICF of $\sqrt{D}$ (which reduce to 5 with the sign convention we use here)

This is for $\sqrt{D}$. But also $(1 + \sqrt{D})/2$ for $D \equiv 1 \pmod{4}$

Williams and Buhr is 5 midpoint criteria under our sign convention; in the appendix
First a few words on symmetries in NSCF of \(\sqrt{D}\).

Type I has the symmetries of the RCF of \(\sqrt{D}\).

In the example, 91 is not a sum of squares = 7 * 13 and is 3 modulo 4

Period length 5

\(a_1 = a_4 = 2\); sum of indexes is 5
\(a_2 = a_3 = 6\)

\(\varepsilon_1 = \varepsilon_5 = -1\); sum of indexes is 6
\(\varepsilon_2 = \varepsilon_4 = +1\)
\(\varepsilon_3 = \varepsilon_3 = -1\)
Type II—same symmetries as Type I except for a different symmetry at the middle of the period.

\[ \xi_3 = \frac{13 + \sqrt{97}}{9} = \frac{9 + 4 + \sqrt{9^2 + 4^2}}{9} \]

so \( p = 9, \; q = 4 \)

period length = 6

\[ a_3 = 2 \]
\[ a_4 = a_2 - 1; \; 2 = 3 - 1 \]

\( \epsilon_3 = -1 \)
\( \epsilon_4 = +1 \)
Mid-Point Criteria for NSCF of $\sqrt{D}$

- Three tests:
  - For Type I—$P$-test and $Q$-test
  - For Type II—$PQ$-test

- $P$-test: If $\widetilde{P}_i = \widetilde{P}_{i+1}$ then $\ell = 2i$
  
  \[ A_{i-1} = A_i B_{i-1} + \varepsilon_i A_{i-1} B_{i-2}, \quad B_{i-1} = B_{i-1} (B_i + \varepsilon_i B_{i-2}) \]
  
  \[ A_{i-1}^2 - DB_{i-1}^2 = 1 \]

From the symmetries, we get three tests. Exactly one of these will hold

read test and note period length

note that we give formulas for $A_{\ell-1}$, $B_{\ell-1}$
Mid-Point Criteria for NSCF of $\sqrt{D}$

- **Q-test:** If $\widetilde{O}_i = \widetilde{O}_{i+1}$, then $\ell = 2i + 1$
  
  $$A_{\ell-1} = A_i B_i + \varepsilon_{i+1} A_{i-1} B_{i-1}$$
  
  $$B_{\ell-1} = B_i^2 + \varepsilon_{i+1} B_{i-1}^2$$
  
  $$A_{\ell-1}^2 - DB_{\ell-1}^2 = -\varepsilon_{i+1}$$

- **PQ-test:** If $\widetilde{P}_i = \widetilde{P}_{i+1} + \frac{1}{2} \widetilde{O}_{i-1}$ and $\varepsilon_i = -1$ then $\ell = 2i$
  
  $$A_{\ell-1} = A_i B_{i-1} - B_{i-2} (A_{i-1} - A_{i-2})$$
  
  $$B_{\ell-1} = 2B_{i-1}^2 - B_i B_{i-2}$$
  
  $$A_{\ell-1}^2 - DB_{\ell-1}^2 = -1$$

Final tests.

If $\sqrt{D}$ meets Q-test, there is not necessarily a unit of negative norm.
NSCF of $\sqrt{137}$

$PQ$-test: $15 = 11 + 8/2$ and $\varepsilon_3 = -1$

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<th>$q_i$</th>
<th>$a_i$</th>
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$P_3 = Q_3 + 0.5Q_2$ and $\varepsilon_3 = -1$ so period length $= 2 \times 3 = 6$

$15 = 11 + 8/2$

$1744^2 - 137 \times 149^2 = -1$

$A_5 = A_3 B_2 - B_1(A_2 - A_1) = 199 \times 10 - 3(117 - 35) = 1990 - 246 = 1744$

$B_5 = 2 B_2^2 - B_3 B_1 = 2 \times 10^2 - 17 \times 3 = 200 - 51 = 149$
Next Up:

NICF-Reduced Quadratic Irrationals

The idea of a reduced irrational is that its CF expansion is purely periodic.
Reduced NICF QI’s

- The NICF for a quadratic irrational $\xi$ is purely periodic if and only if $\xi > 2$ and the conjugate of $\xi$ satisfies
  
  $$-0.618 = \frac{1 - \sqrt{5}}{2} < \xi \leq \frac{3 - \sqrt{5}}{2} = 0.382$$

- Hurwitz[4] proves this; our proof is more direct, self-contained, and makes minimal use of Hurwitz’ singular continued fractions.

Of course, the bar indicates conjugate

For RCF, $\xi > 1$ and $-1 < \xi-bar < 0$.

For NSCF, more complicated (see appendix); “special”, “semi-reduced”, “reduced” — Maybe reader can find something better!

Special: $Q_v^2 + (1/4)Q_{v+1}^2 \leq D$ and $Q_{v+1}^2 + (1/4)Q_v^2 \leq D$
An example

\[ \xi > 2, \quad -0.618 < \bar{\xi} < 0.382 \]

First reduced is \( \xi_3 \); all subsequent are reduced
Questions?
References


The Ayyangar papers are available at
http://www.ms.uky.edu/~sohum/AAK/PRELUDE.htm
References


References

Appendix
Midpoint Criteria for NICF

1. $P_{i+1} = P_i, \quad \ell = 2i$
2. $P_{i+1} = P_i + Q_i, \quad \ell = 2i$
3. $Q_{i+1} = Q_i$ and $\varepsilon_{i+1} = -1, \quad \ell = 2i + 1$
4. $Q_{i+1} = Q_i$ and $\varepsilon_{i+1} = +1, \quad \ell = 2i + 1$
5. $P_{i+1} = Q_i + (Q_{i+1})/2, \quad \ell = 2i + 1$
6. $P_{i+1} = (Q_i)/2 + Q_{i+1}, \quad \ell = 2(i+1)$

3 and 4 can be combined under this sign convention
Comparison of Tests

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<th>Type of CF</th>
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<td>NSCF</td>
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</table>

Count is number of occurrences for $\sqrt{D}$, $D$ not a square, up to 25 million, in millions

NICF case 1 is the overwhelmingly most popular (occurs nearly 80% of the time)

Note for RCF, Q test occurs about 10% of the time
### Comparison of NSCF and RCF periods for $\sqrt{D}$.

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<td>2,604,125,007</td>
<td>3,751,067,951</td>
<td>0.6942356</td>
</tr>
<tr>
<td>8,000,000</td>
<td>3,165,696,279</td>
<td>4,559,939,520</td>
<td>0.6942408</td>
</tr>
<tr>
<td>9,000,000</td>
<td>3,606,639,205</td>
<td>5,416,886,128</td>
<td>0.694243</td>
</tr>
<tr>
<td>10,000,000</td>
<td>4,387,213,325</td>
<td>6,319,390,242</td>
<td>0.6942463</td>
</tr>
</tbody>
</table>

$I\Lambda(n)$ is the sum of the NSCF period lengths of $\sqrt{D}$ up to $n$, $D$ not a square

$P(n)$ is the same for RCF

$log((1 + \sqrt{5})/2)/log(2) = 0.6942419136 ...$

$log((1 + \sqrt{5})/2)/log(2) = 0.6942419136 ...$

0.69424191363061730173879026689859522346356728522712971598098986654140574
4105011761897631417234764535973442
Reduced QI for RCF and NSCF

- A quadratic irrational $\xi_0$ is RCF-reduced if
  1. $\xi_0 > 1$
  2. $-1 < \xi_0 < 0$

- For NSCF
  - A quadratic irrational $\xi_0$ is "special" if
    $$\frac{Q_0^2}{4} + \frac{Q_1^2}{4} \leq D,$$
    $$\frac{Q_1^2}{4} + \frac{Q_0^2}{4} \leq D$$
  - The successor of a special surd is "semi-reduced"
  - The successor of a semi-reduced surd is "reduced"

- A reduced surd always has a purely periodic continued fraction expansion; special and semi-reduced surds do not necessarily have purely periodic continued fraction expansions
One Step of NICF or NSCF
Can be Two Steps of RCF

- Let $x =$ fractional part of $\xi_i$, assume that $0.5 < x < 1.0$. Then:
- For RCF
  - $1 < \frac{\xi}{s_{i+1}} = 1/x < 2$,
  - $a_{i+1} = 1$, and
  - $\frac{\xi}{s_{i+2}} = 1/(\frac{\xi}{s_{i+1}} - a_{i+1}) = 1/(1/x - 1) = x/(1-x)$
- For NICF, $\xi_{i+1} = 1/(1-x)$
- But $1/(1-x) - x/(1-x) = 1$

Does the algebra to show that if $a_{i+1} = 1$ then one NICF step is 2 RCF steps

Also, if $a_{i+1} = 1$ for RCF, then $x > 0.5$

$\xi_i$ does not have to be a quadratic irrational for the above