THEORY OF 4×4 MAGIC SQUARES

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Abstract. We give some properties of general 4×4 magic squares.

1. Introduction

A 4×4 magic square is a constant called the *magic sum* and a 4×4 array with 16 distinct entries so that every row, every column, and the two main diagonals each add to the magic sum. Here's a magic square with magic sum 188:

$$\begin{array}{rrrr} 127 & 33 & 7 & 21 \\ 51 & 27 & 93 & 17 \\ 1 & 65 & 3 & 119 \\ 9 & 63 & 85 & 31 \end{array}$$

Magic squares can have additional properties. For example, in the square above, every entry when written as a binary number is a palindrome.

There are some other sums that are also always the magic sum, S. For the arbitrary 4×4 magic square

$$\begin{array}{cccc} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & O & P \end{array}$$

we always have that

$$A + D + M + P = F + G + J + K = S$$

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and

$$B + C + N + O = E + I + H + L = S.$$

Given a magic square, any rotation or reflection gives another magic square. Any square is one of a set of eight that are equivalent under rotation or reflection.

There are two other transforms that always produce magic squares from a given magic square. The first, which we will call $T1$, is to switch the second and third rows, and then to switch the second and third columns. Starting with the square above, this results in

$$\begin{array}{cccc} A & C & B & D \\ I & K & J & L \\ E & G & F & H \\ M & O & N & P \end{array}$$

The second transform, which we will call $T3$, is to switch the first and second columns, switch the third and fourth columns, switch the first and second rows, and switch the third and fourth rows. This gives:

$$\begin{array}{cccc} F & E & H & G \\ B & A & D & C \\ N & M & P & O \\ J & I & L & K \end{array}$$

Using $T1$, $T3$, and rotations and reflections gives 32 squares equivalent to any given square. This isn't completely obvious because the transforms are not necessarily commutative.

Note that switching the first and fourth columns, and switching the first and fourth rows gives the same result as $T1$ rotated 180 degrees.

Because of the above, when searching for 4×4 magic squares, you can insist that

$$A > S/4,$$

$$A > D > M,$$

$$A > P,$$

$$B > C,$$

and that A is greater than any of F, G, J, or K.

This can speed up the search process.

To search, make the outermost loop a loop over the magic sum, MS. Loop over 7 entries in the square (the entries plus the magic sum give 17 variables; there are 9 independent equations–4 rows, 3 columns, 2 diagonals). For each of 7 to loop over, only go up to a limit determined from magic sum (which we loop over in the outermost loop) and other entries already set. Loop on MS, $n11$, $n12$, $n13$, $n41$, $n22$, $n23$, $n21$ (nrc is the entry for row r and column c). Deduce as many as possible along the way.

Each new n should be positive, not equal to any previously set, meet any properties wanted (binary palindrome, square,). Sometimes when a new n is negative, we go back to previous loop; sometimes continue current loop. Ditto if n is too big (e.g., n is one of the central 4 and has to be less than the upper-leftmost entry).

More details are given in GEN4x4.txt

2. Group of Transforms

Define some transforms as follows:

T1 switches midddle two columns, middle two rows

T2 switches top and bottom rows, leftmost and rightmost columns

T3 switches rows 1 and 2, 3 and 4, and columns 1 and 2, 3 and 4

TR transpose, leaving main diagonal in place

R1 rotate 90 degrees clockwise (Rn is rotate by 180 n degrees)

ID Does nothing

Then T1, T3, R1, and TR generate the group of transforms, and no subset of these generates the group. Every element of the group can be written as T1^a T3^b R1^c TR^d with $0 \leq a, b, d \leq 1$ and $0 \leq c \leq 3$. It is straightforward to check that the $32 = 2 \times 2 \times 4 \times 2$ elements are distinct. To see that all 32 give magic squares, it suffices to check that each of T1, T3, R1, and TR give a magic square. To see that there are no transforms other than these 32 that always give a magic square, it suffices to do a brute force search on possible arrangements of certain sets of 16 numbers and see that the only ways they make a magic square is the 32 these transformations give.

The following relations suffice to reduce any string of transformations to one of canonical form.

T1 and T3 commute with R1 and TR
T3 T1 = T1 T3 R2
TR R1 = R3 TR
TR R2 = R2 TR
TR R3 = R1 TR
T2 = T1 R2

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